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Almost unbounded L and M-weakly compact operators

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Abstract. In this paper, we introduce and investigate a new class of operators known as almost unbounded L-weakly compact (in shortly, ${}_{au}L$-weakly compact) and almost unbounded M-weakly compact (in shortly, ${}_{au}M$-weakly compact) operators. We explore the lattice properties related to this class and examine their relationships with other established operator classes, such as L-weakly compact operators and almost L-weakly compact operators. We demonstrate that every L-weakly compact operator is an ${}_{au}L$-weakly compact operator, but the reverse implication does not necessarily hold in all cases.

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1. Introduction and preliminaries

The notion of unbounded order convergence (uo-convergence) was initially introduced by Nakano [11] in the context of vector lattices. Subsequently, the authors in a separate study [3] introduced and investigated unbounded positive operators on vector spaces. The purpose of this article is to develop lattice and topological concepts that are specifically designed for unbounded states. We thoroughly investigate the properties of these newly introduced concepts and explore their connection with the bounded case. For more in-depth information on some of these extensions, please refer to the references [7] and [8]. This paper introduces the space of almost unbounded L-weakly compact (in shortly, ${}_{au}L$-weakly compact) and almost unbounded M-weakly compact (in shortly, ${}_{au}M$-weakly compact) operators for the first time. These operators are broader than the class of almost L-weakly (resp. M-weakly) compact operators, which was initially introduced by Bouras in [4]. The class of unbounded L-weakly (resp. M-weakly) compact operators was subsequently

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introduced by Niktab et al., in [12]. These definitions are based on a new concept known as L-weakly (resp. M-weakly) compact operators, which was introduced by Meyer-Nieberg in [10].

In this paper, we adopt the following notations:

1. X and Y represent real Banach spaces.
2. E and F represent real Banach lattices.
3. B_X denotes the closed unit ball of X.
4. $sol(A)$ denotes the solid hull of a subset A in a Banach lattice.
5. The term "operator" between two Banach spaces is defined as a bounded linear mapping.

Additionally, we use the notation c to refer to the Banach lattice of all convergent sequences, while c_0 represents the Banach lattice of all null convergent sequences. A net (x_α) in E is unbounded norm convergent or un-convergent to x if for each $u \in E_+$, $\||x_\alpha - x| \wedge u\| \to 0$, we then write $x_n \ un \longrightarrow x$. A Banach lattice E is said to be an AM-apace if for each $x, y \in E$, we have $|x + y| = max\{|x|, |y|\}$ where $|x| \wedge |y| = 0$. Recall that a norm $\|.\|$ of a Banach lattice E is order continuous if for each net (x_α) in E with $x_\alpha \downarrow 0$, one has $\|x_\alpha\| \downarrow 0$. By Theorem 4.9 of [1], it is easy to see that every reflexive Banach lattice has order continuous norm. For example, we can observe that the Banach lattice l_2 possesses an order-continuous norm. A Banach lattice is said to have weakly sequentially continuous lattice operations whenever $x_n \ w \longrightarrow 0$ implies $|x_n| \xrightarrow{w} 0$. Every AM-space and every Banach space with Schur property has this property. As an example we can see that l_∞ has a weakly sequentially operations.

An operator T from E into Y is called Dunford-Pettis operator, if the sequence $(T(x_n))$ converges to 0 for every weakly null sequence (x_n) of E. An operator T from E into Y is almost Dunford-Pettis, if the sequence $(T(x_n))$ converges to 0 for every weakly null sequence (x_n) of E consisting of pairwise disjoint elements, equivalently, $(T(x_n))$ converges to 0 for every weakly null sequence (x_n) of E_+ from [4]. Recall that a continuous operator $T : X \longrightarrow E$ from a Banach space to a Banach lattice is said to be semi-compact if and only if for each $\varepsilon > 0$ there exists some $u \in E_+$ such that $T(B_X) \subset [-u, u] + \varepsilon B_E$.

A nonempty bounded subset A of E is said to be L-weakly compact if $\lim \|x_n\| = 0$ for every disjoint sequence (x_n) contained in the solid hull of A. A nonempty bounded subset A of E is said to be unbounded L-weakly compact if $\lim \||x_n| \wedge u\| = 0$ for every disjoint sequence (x_n) contained in the solid hull of A and all $u \in E_+$, equivalently, $\lim \|x_n\| = 0$ for every order bounded disjoint sequence (x_n) contained in the solid hull of A.

Example 1. Since every order continuous Banach lattice is unbounded L-weakly compact, both B_{l_1} (the closed unit ball of l_1) and B_{c_0} (the closed unit ball of c_0) are unbounded L-weakly compact subsets of l_1 and c_0, respectively.

An operator T from X into F is said to be L-weakly compact if $T(B_X)$ is a L-weakly compact subset of F.

Similarly, an operator T from E into Y is called M-weakly compact if $\lim \|T(x_n)\| = 0$ holds for every norm-bounded disjoint sequence (x_n) in E.

In fact, the paper [12] provides a classification of unbounded L-weakly compact operators (referred to as u-L-weakly compact operators) and unbounded M-weakly compact operators (referred to as u-M-weakly compact operators). According to the definition, an operator T from X to F is considered unbounded L-weakly compact (u-L-weakly compact) if $T(B_X)$ is an unbounded L-weakly compact subset of F. Similarly, an operator T is considered unbounded M-weakly compact (u-M-weakly compact) if $T(B_X)$ is an unbounded M-weakly compact subset of F.

It is worth noting that every set in an order continuous Banach lattice is unbounded L-weakly compact. This is due to Meyer-Nieberg's theorem, which states that every disjoint order-bounded sequence in an order continuous Banach lattice is norm null. As a result, every operator acting into an order continuous Banach lattice is unbounded L-weakly compact.

Based on Example 1 in the paper [12], it is demonstrated that the identity operators on ℓ_1 and c_0 serve as examples of such operators. This is due to the fact that both c_0 and ℓ_1 have order continuous norms, and every disjoint sequence in c_0 and ℓ_1 is norm null. Let's consider the specific inclusion operator $T : \ell_1 \to c_0$. For any norm-bounded disjoint sequence (a_n) in ℓ_1, and for any $u \in c_0^+$, the sequence $(a_n \wedge u)$ is disjoint in c_0. By utilizing the order continuity of c_0, we can conclude that $a_n \wedge u$ converges to 0. Additionally, any norm-bounded disjoint sequence in the solid hull of $T(B_{\ell_1})$ converges to 0 in norm. Therefore, T is also u-L-weakly compact.

The canonical inclusion map from l_1 to c_0 serves as an example of an operator that is unbounded M-weakly compact, meaning it maps relatively weakly compact sets to unbounded M-weakly compact sets. However, it is not M-weakly compact, indicating that it does not map weakly compact sets to M-weakly compact sets.

Let's consider the canonical inclusion operator $T : l_1 \longrightarrow c_0$. Firstly, we observe that T is not L-weakly compact because $|T(e_n)| = |e_n| = 1$ for each n, where (e_n) represents the standard unit vector basis of l_1.

Next, let (a_n) be a norm-bounded disjoint sequence in l_1. Since (a_n) is also a norm-bounded disjoint sequence in c_0, and B_{c_0} is unbounded L-weakly compact, we have $\lim ||T(a_n)| \wedge u| = \lim ||a_n| \wedge u| = 0$ for each $u \in (c_0)_+$. Therefore, T is u-M-weakly compact.

On the other hand, every norm-bounded disjoint sequence in the solid hull of $T(B_{l_1})$, which is a subset of B_{c_0}, is unbounded and converges to zero in norm. Hence, T is also u-L-weakly compact.

In summary, the canonical inclusion operator $T : l_1 \longrightarrow c_0$ is not L-weakly compact, but it is both u-M-weakly compact and u-L-weakly compact.

An operator T from a Banach space X into a Banach lattice F is called almost L-weakly compact if T carries relatively weakly compact subsets of X onto L-weakly compact subsets of F. Similarly, an operator T from a Banach lattice E into a Banach space Y is called almost M-weakly compact if for every disjoint sequence (x_n) in B_E and every weakly convergent sequence (f_n) of Y', we have $f_n \circ T(x_n) \to 0$, for more information see [4].

An operator $T : E \longrightarrow F$ between two vector lattices is called disjointness preserving if $Tx \perp Ty$ for all x, $y \in E$ satisfying $x \perp y$. By Meyer's theorem (Theorem 3.14) from [10], we know that, if an order bounded operator $T : E \longrightarrow F$ between two Archimedean vector lattices preserves disjointness, then its modulus exists, and

$$|T|(|x|) = |T(|x|)| = |Tx|,$$

holds for all $x \in E$. Finally, we conclude this paper by providing a sufficient condition under which, for each auL-weakly compact (resp. auM-weakly compact) operator T, $|T|$ is a auL-weakly compact (resp. auM-weakly compact) operator.

In the subsequent sections of this article, the following concepts and symbols will be utilized:

$L\,(X\,,\,Y)$ will denote the space of all operators from X into Y, $L\,W\,(X\,,\,F)$ will denote the space of all L-weakly compact operators from X into F, $M\,W\,(E\,,\,Y)$ will denote the space of all M-weakly compact operators from E into Y, $L\,W_u\,(X\,,\,F)$ will denote the space of all unbounded L-weakly compact operators from X into F, $M\,W_u\,(E\,,\,F)$ will denote the space of all unbounded M-weakly compact operators from E into F, $L\,W_{au}\,(X\,,\,F)$ will denote the space of all ${}_{au}L$-weakly compact operators from X into F, $M\,W_{au}\,(E\,,\,Y)$ will denote the space of all ${}_{au}M$-weakly compact operators from E into Y.

To gain a better understanding of the terminology and concepts related to Banach lattices and positive operators, we recommend consulting the monographs by Aliprantis [1] and Meyer [9]. These references provide comprehensive coverage of the subject matter and can serve as valuable resources for further exploration.

2. Main results

A Banach lattice E is said to have the positive unbounded Schur property if weakly null sequences with positive terms are unbounded in norm, or equivalently, if every order bounded weakly null sequence with positive terms is norm-null. It is clear that if a Banach lattice E possesses the positive Schur property, then it also possesses the positive unbounded Schur property. However, the converse does not hold in general.

Example 2. (a) The Banach lattice c_0 possesses the positive unbounded Schur property, but does not have the positive Schur property.

(b) According to Proposition 6.3 in [5], every Banach lattice with an order continuous norm possesses the positive unbounded Schur property. As a result, the Banach lattice l_2 (the space of square-summable sequences) has the positive unbounded Schur property. However, l_2 does not have the positive Schur property.

(c) Since l_∞ (the space of bounded sequences) is not order continuous, it does not possess the positive unbounded Schur property.

The positive unbounded Schur property is equivalent to the Banach lattice being order continuous. This result can be derived from Theorem 4.17 in [1] or Proposition 6.3 in [5], which state that every positive weakly null sequence in an order continuous Banach lattice is order continuous. Conversely, if a Banach lattice has the positive unbounded Schur property, we can observe that every order bounded disjoint positive sequence is weakly null. Therefore, by the assumption of the positive unbounded Schur property, these sequences are norm null. By Meyer-Nieberg's Theorem, this implies order continuity.

Definition 1. An operator T from a Banach space X into a Banach lattice F is said to be almost unbounded L-weakly compact (denoted as ${}_{au}L$-weakly compact) if T maps relatively weakly compact subsets of X onto unbounded L-weakly compact subsets of F.

It is well-known that every L-weakly compact operator is also ${}_{au}L$-weakly compact. This can be easily demonstrated by observing that every weakly compact set is necessarily norm bounded. However, it should be noted that the converse does not hold in general, indicating that not every ${}_{au}L$-weakly compact operator is necessarily L-weakly compact. For instance, consider the identity operator $Id_{l_1} : l_1 :\longrightarrow l_1$. Since the Banach space l_1 has the Schur property, it follows from Corollary 3.6.8 in [9] that every relatively weakly compact subset of l_1 is L-weakly compact. Therefore, Id_{l_1} is ${}_{au}L$-weakly compact. On the other hand B_{l_1} is not relatively weakly compact, and therefore is not L-weakly compact. Hence Id_{l_1} is not L-weakly compact.

On the other hand, it is true that every almost L-weakly compact operator is ${}_{au}L$-weakly compact. This follows from the observation that every set that converges in norm is also unbounded norm convergent. However, the converse does not hold in general. As an example, let's consider the operator $Id_{c_0} : c_0 \longrightarrow c_0$. Since the Banach space c_0 has the positive unbounded Schur property, it can be concluded that Id_{c_0} is ${}_{au}L$-weakly compact. However, c_0 does not have the positive Schur property. Therefore, according to Proposition 2.2 in [4], it follows that Id_{c_0} is not almost L-weakly compact.

Definition 2. An operator T from a Banach lattice E into a Banach space Y is called almost unbounded M-weakly compact (in shortly, ${}_{au}M$-weakly compact) if for every disjoint sequence (x_n) in B_E and every weakly convergent sequence (f_n) of Y' and all $g \in (E')_+$, we have $(|f_n \circ T| \wedge g)(x_n) \to 0$.

Example 3. Consider the space $E = l_2$. According to Theorem 4.9 in [1], E possesses an order continuous norm. Additionally, Proposition 6.3 from [5] implies that E has the positive unbounded Schur property. However, it is known that E does not satisfy the positive Schur property. Consequently, we can deduce that the identity operator $I : l_2 \to l_2$ is ${}_{au}M$-weakly compact. On the other hand, Corollary 2.1 in [4] states that it is not almost M-weakly compact.

It is well-established that the classes of L-weakly compact operators and M-weakly compact operators exhibit a duality relationship. Specifically, an operator T between two Banach lattices is L-weakly compact if and only if its adjoint T' is M-weakly compact. This duality relationship is established in Proposition 3.6.11 of [9]. In the upcoming discussion, we will establish a similar result for the classes of ${}_{au}L$-weakly compact operators and ${}_{au}M$-weakly compact operators. We will make use of several lemmas throughout this paper. The first lemma corresponds to a specific case of Theorem 5.63 in [1], while the subsequent lemmas are derived from the same theorem.

Lemma 1. *Let $A \subset E$ and $B \subset E'$ be nonempty and bounded, and let $u \in E_+$. For each disjoint sequence (x_n) in the solid hull of A, $sup\{|f(|x_n| \wedge u)| : f \in B\} \to 0$ if and only if for each disjoint sequence (f_n) in the solid hull of B,*

$$sup\{|f_n(|x| \wedge u)| : \ x \in A\} \to 0.$$

Lemma 2. *Let $A \subset E$ and $B \subset E'$ be nonempty and bounded, and let $f \in (E')_+$. For each disjoint sequence (x_n) in the solid hull of A, $sup\{|(|g| \wedge f)(x_n)| : \ g \in B\} \to 0$ if and only if for each disjoint sequence (f_n) in the solid hull of B,*

$$sup\{|(|f_n| \wedge f)(x)| : \ x \in A\} \to 0.$$

Lemma 3. *For every nonempty bounded subset $A \subset E'$, every sequence (x_n) of B_E, and $f \in (E')_+$, the following assertions are equivalent*

(1) *$sup\{|(|g| \wedge f)(x_n)| : \ g \in A\} \to 0$.*
(2) *For each sequence (f_n) of A, $(|f_n| \wedge f)(x_n) \to 0$.*

Proposition 1. *For a Banach lattice E the following assertions hold.*

(i) *If $A \subset E$ is nonempty bounded subset, then the following statements are equivalent.*
   (1) *A is unbounded L-weakly compact.*
   (2) *$f_n(|x_n| \wedge u) \to 0$ for every sequence (x_n) of A, every disjoint sequence (f_n) of $B_{E'}$ and all $u \in E_+$.*
   (3) *$f_n(x_n) \to 0$ for every order bounded sequence (x_n) of $sol(A)$, every disjoint sequence (f_n) of $B_{E'}$.*

(ii) *If $A \subset E'$ is nonempty bounded subset, then the following statements are equivalent.*
   (4) *A is unbounded L-weakly compact.*
   (5) *$(|f_n| \wedge f)(x_n) \to 0$ for every sequence (f_n) of A, every disjoint sequence (x_n) of B_E and all $f \in (E')_+$.*
   (6) *$f_n(x_n) \to 0$ for every order bounded sequence (f_n) of $sol(A)$, every disjoint sequence (x_n) of B_E.*

Proof.

(1) $\Leftrightarrow$ (2) Let A be a nonempty bounded subset of E. A is unbounded L-weakly compact if and only if $\||x_n| \wedge u\| \to 0$ holds for each disjoint sequence (x_n) of $sol(A)$ and all $u \in E_+$. Thus A is unbounded L-weakly compact if and only if for every disjoint sequence (x_n) of $sol(A)$ and all $u \in E_+$,

$$sup\{|f(|x_n| \wedge u)| : \ f \in B_{E'}\} \to 0.$$

By Lemma 1, this is equivalent to saying that for every disjoint sequence (f_n) of $B_{E'}$ and all $u \in E_+$,

$$sup\left\{\left|f_n\left(|x| \wedge u\right)\right| : \ x \in A\right\} \rightarrow 0.$$

(2) $\Leftrightarrow$ (3) Obvious.
(4) $\Leftrightarrow$ (5) We can utilize Lemma 3 to establish a proof similar to that of (i).
(5) $\Leftrightarrow$ (6) Obvious.

□

The following result provides a sequential characterization of almost unbounded L-weakly compact operators.

Similarly, we can establish the following result, and its proof exhibits similarities to Proposition 1.

Theorem 1. *An operator φ from a Banach space X' into a Banach lattice F' is ${}_{au}L$-weakly compact, if and only if, $\left(\left|\varphi\left(f_n\right)\right| \wedge g\right)\left(y_n\right) \rightarrow 0$ for every weakly convergent sequence (f_n) of X', every disjoint sequence (y_n) of B_F and for all $g \in (F')_+$.*

Theorem 2. *For an operator T from a Banach space X into a Banach lattice F the following assertions are equivalent.*

1. *T is ${}_{au}L$-weakly compact.*
2. *$y_i \rightarrow 0$ for every order bounded disjoint sequence (y_i) contained in the solid hull of $T(A)$ where $A \subseteq X$ is relatively weakly compact set.*
3. *$y_i \rightarrow 0$ for every order bounded disjoint sequence (y_i) contained in the solid hull of $\{T\left(x_n\right) : n \in \mathbb{N}\}$ where $(x_n) \subseteq X$ is relatively weakly compact sequence.*
4. *$y_i \rightarrow 0$ for every order bounded disjoint sequence (y_i) contained in the solid hull of $\{T\left(x_n\right) : n \in \mathbb{N}\}$ where $(x_n) \subseteq X$ is weakly convergent sequence in X.*

Proof.

(1) $\Leftrightarrow$ (2) Obvious.

(1) $\Rightarrow$ (3) Let $(x_n) \subseteq X$ be a relatively weakly compact sequence and (y_i) be an order bounded disjoint sequence contained in the solid hull of $\{T(x_n) : n \in \mathbb{N}\}$. Then there exists $w \in F_+$ that $|y_i| \leq w$ for all $i \in \mathbb{N}$. It follows that $\|y_i\| = \||y_i| \wedge w\| \rightarrow 0$.

(3) $\Rightarrow$ (1) Consider an operator T from X into F. Let A be relatively weakly compact and (y_n) be a disjoint sequence contained in the solid hull of $T\left(A\right)$. Then there exists $(x_n) \subseteq A$ such that $|y_n| \leq |T(x_n)|$ for all $n \in \mathbb{N}$. Since $(x_n) \subseteq A$, then (x_n) is relatively weakly compact. The sequence $(|y_n| \wedge w)$ is an order bounded disjoint sequence of F for all $w \in F_+$. By apply (2), $|y_n| \wedge w \rightarrow 0$ for all $w \in F_+$. Hence T is ${}_{au}L$-weakly compact.

(3) $\Rightarrow$ (4) Let T be an operator from X into F, and let (x_n) be a weakly convergent sequence in X. It follows that (x_n) is relatively weakly compact. Now, consider an order bounded disjoint sequence (y_i) contained in the solid hull of $\{T\left(x_n\right) :, n \in \mathbb{N}\}$, we have $y_i \rightarrow 0$.

(4) $\Rightarrow$ (3) Let (x_n) be relatively weakly compact sequence of X, and let (y_i) be an order bounded disjoint sequence of $sol\{\{T(x_n) : n \in \mathbb{N}\}\}$. Then there exists $n_i \in \mathbb{N}$ such that $|y_i| \leq |T(x_{n_i})|$ for all $i \in \mathbb{N}$. Assume by way of contradiction that $y_i \nrightarrow 0$, then there exists $\varepsilon > 0$ and a subsequence $\left(x_{\varphi(n_i)}\right)$ of (x_{n_i}) and $(y_{\phi(i)})$ satisfying

$$y_{\varphi(i)} > \varepsilon$$

for all $i \in \mathbb{N}$. The sequence $\left(x_{\varphi(n_i)}\right)$ is relatively weakly compact, then there exists a weakly convergent subsequence $\left(x_{\psi(\varphi(n_i))}\right)$ of $\left(x_{\varphi(n_i)}\right)$ in X such that

$$y_{\psi(\varphi(i))} \to 0,$$

which is absurd. So $y_i \to 0$, and the proof follows.

$\square$

Proposition 2. *For a Banach lattice E and a Banach space X the following statements hold.*

1. *The set of all ${}_{au}L$-weakly compact operators from X into E is a closed subspace of $L(X, E)$.*
2. *The set of all ${}_{au}M$-weakly compact operators from E into X is a closed subspace of $L(E, X)$.*

Proof. (1) Let $T_1, T_2 \in LW_{au}\,(X, E)$ and let A be relatively weakly compact subset of X. Assume that (y_n) is an order bounded disjoint sequence contained in the solid hull of $(T_1 + T_2)(A)$. By Decomposition property, Theorem 1.13 from [1], there exists the positive terms sequence $(y_{i,n})$ contained in the solid hull of $T_i(A)$ for $i = 1, 2$ such that $|y_n| = y_{1,n} + y_{2,n}$ for each n. Clearly, $(y_{i,n})$ is an order bounded disjoint sequence for $i = 1, 2$. Since T_1, T_2 are ${}_{au}L$-weakly compact, by Theorem 2, we have $y_{i,n} \to 0$, for $i = 1, 2$. It follows that $y_n \to 0$. We conclude from Theorem 2, $T_1 + T_2 \in LW_{au}(X, E)$.

Let $T \in \overline{LW_{au}(X, E)}$. Let $\varepsilon > 0$ and let $S \in LW_{au}\,(X, E)$ such that $\|T - S\| < \varepsilon$ and A is relatively weakly compact subset of X. Assume that (y_n) contained in the solid hull of $T(A)$ be an order bounded disjoint sequence. For each $n \in \mathbb{N}$ choose some $u_n \in A$ such that $|y_n| \leq |T(u_n)|$. We have

$$0 \leq |y_n| \leq |(T - S)(u_n)| + |S(u_n)|.$$

It follows from Theorem 1.13 in [1] that for each $n \in \mathbb{N}$ there exists $v_n, z_n \in E_+$ with $v_n \leq |(T - S)(u_n)|$ and $z_n \leq |S(u_n)|$ such that $|y_n| = v_n + z_n$. Clearly, (z_n) is an order bounded disjoint sequence in $sol(S(A))$. Since S is ${}_{au}L$-weakly compact, by Theorem 2, $z_n \to 0$. Thus we see that $y_n \to 0$ holds, as desired.

(2) Let $T \in \overline{MW_{au}(E, X)}$, and let (x_n) be a disjoint sequence of B_E and (f_n) be a weakly convergent sequence of X', and $g \in (X')_+$. Without loss of generality, (x_n) is positive term. Let $\varepsilon > 0$. There exists some

${}_{au}M$-weakly compact operator $S : E \longrightarrow X$ such that $\|T - S\| < \varepsilon$. We have

$$(|f_n \circ T| \wedge g)(x_n) = (|f_n \circ (T - S) + f_n \circ S| \wedge g)(x_n)$$

$$\leq \varepsilon \|f_n\| + (|f_n \circ S| \wedge g)(x_n),$$

where $\limsup (|f_n \circ T| \wedge g)(x_n) \leq \varepsilon . M$. Then proof follows. □

Proposition 3. *For a Banach lattice E the following statements are equivalent.*

(1) $Id_E \in LW_{au}(E)$.
(2) $f_n(|x_n| \wedge u) \to 0$ *for every weakly convergent sequence* (x_n) *of* E, *every disjoint sequence* (f_n) *of* $B_{E'}$ *and all* $u \in E_+$.
(3) *E has the positive unbounded Schur property.*

Proof. (1) ⇔ (2) This follows from Theorem 1.

(2) ⇒ (3) Let (x_n) be a weakly null sequences with positive terms. Then

$$f_n(|x_n| \wedge u) \to 0,$$

for every disjoint sequence (f_n) of $B_{E'}$ and $u \in E_+$. It follows that $0 \leq x_n \wedge u \leq x_n$ for each n. This yields $x_n \wedge u \xrightarrow{w} 0$. By Corollary 2.6 in [6], we have $|x_n| \wedge u \to 0$ for all $u \in E_+$. Then E is positive unbounded Schur property.

(3) ⇒ (1) Let (x_n) be a weakly convergent sequence of E. If (y_i) be disjoint sequence contained in the solid hull of $\{x_n : n \in \mathbb{N}\}$, the set $\{x_n : n \in \mathbb{N}\}$ is relatively weakly compact, then every disjoint sequence in $A = sol(\{x_n : n \in \mathbb{N}\})$ converges weakly to zero, hence $|y_i| \wedge u \to 0$ for all $u \in E_+$. By Theorem 2, the proof follows. □

Corollary 1. *For a Banach lattice E the following statements are equivalent.*

(1) *E' has the positive unbounded Schur property.*
(2) $Id_{E'} \in LW_{au}(E')$.
(3) $(|f_n| \wedge f)(x_n) \to 0$ *for every weakly convergent sequence* (f_n) *of* E', *disjoint sequence* (x_n) *of* B_E *and all* $f \in (E')_+$.
(4) $Id_E \in MW_{au}(E)$.

Theorem 3. *For an operator T from X into F, the following statements are equivalent:*

(1) *T is ${}_{au}L$-weakly compact.*
(2) *If $S : Z \longrightarrow X$ is weakly compact operator from an arbitrary Banach space Z into X, the product $T \circ S$ is unbounded L-weakly compact.*
(3) *If $S : l_1 \longrightarrow X$ is weakly compact operator, the product $T \circ S$ is unbounded L-weakly compact.*

Proof. (1) ⇒ (2) and (2) ⇒ (3) Obvious.

(3) $\Rightarrow$ (1) According to Theorem 1, it suffices to show that $f_n(|T(x_n)| \wedge w) \to 0$ for every weakly convergent sequence (x_n) of X, every disjoint sequence (f_n) of $B_{F'}$ and all $w \in F_+$. Indeed, let (x_n) and (f_n) be such sequences. Consider the operator $S : l_1 \to X$ defined by:

$$\forall\, (\alpha_n) \in l_1,\ S\left((\alpha_n)\right) = \sum \alpha_n x_n.$$

Let (e_n) denote the sequence of basic unit vectors of l_1. In this context, we have $S\,(e_n) = x_n$. By Theorem 5.26 of [1], S is weakly compact, and by hypothesis, the product operator $T \circ S$ is unbounded L-weakly compact. Since (e_n) is a norm bounded sequence in l_1, by Theorem 1,

$$f_n\left(|T \circ S\,(e_n)\,| \wedge w\right) \to 0.$$

Hence $f_n\left(|T\,(x_n)\,| \wedge w\right) = f_n\left(|T \circ S\,(e_n)\,| \wedge w\right) \to 0$, as desired. □

Corollary 2. *For a Banach lattice E the following statements are equivalent.*

(1) *E has the positive unbounded Schur property.*
(2) *For each Banach space X, every weakly compact operator $T : X \longrightarrow E$ is unbounded L-weakly compact.*
(3) *Every weakly compact operator $T : l_1 \longrightarrow E$ is unbounded L-weakly compact.*

The subsequent theorem follows directly from the definition of ${}_{au}M$-weakly compact operator, so we omit its proof.

Theorem 4. *An operator $T : E \longrightarrow X$ is ${}_{au}M$-weakly compact if and only if $T' : X' \longrightarrow E'$ is ${}_{au}L$-weakly compact.*

It is clear that every M-weakly compact operator is also ${}_{au}M$-weakly compact. This follows from the fact that if T is an M-weakly compact operator, then its adjoint T' is an L-weakly compact operator. Consequently, T' is also ${}_{au}L$-weakly compact by Theorem 4. Therefore, T is ${}_{au}M$-weakly compact.

However, the converse is not generally true. To illustrate this, let's consider the operator Id_{c_0}. Since $c_0' = l_1$ possesses the positive unbounded Schur property, according to Corollary 1, Id_{c_0} is ${}_{au}M$-weakly compact. On the other hand, Id_{l_1} is not L-weakly compact, and as a consequence, Id_{c_0} is not M-weakly compact.

According to Theorem 2.5 in [4], if T is an almost M-weakly compact operator, then its adjoint T' is an almost L-weakly compact operator. Consequently, T' is also a ${}_{au}L$-weakly compact operator. Therefore, by applying Theorem 4, we can conclude that T is a ${}_{au}M$-weakly compact operator.

Remark 1. It should be noted that if T is a ${}_{au}L$-weakly compact operator, it does not necessarily imply that T' is also ${}_{au}M$-weakly compact. For example, according to Proposition 3, the identity operator on the space c_0 is ${}_{au}L$-weakly compact (since the Banach lattice c_0 has the positive unbounded Schur property). However, by Corollary 1, $Id_{l_1} = (Id_{c_0})'$ is not ${}_{au}M$-weakly compact (as the Banach lattice $(l^1)' = l_\infty$ does not have the positive unbounded Schur property).

Theorem 5. *For an operator T from E into Y, the following statements are equivalent:*

1. *T is ${}_{au}M$-weakly compact.*
2. *If $S: Y \longrightarrow Z$ is weakly compact operator from Y into an arbitrary Banach space Z, the product $S \circ T$ is ${}_{au}M$-weakly compact.*
3. *If $S: Y \longrightarrow c_0$ is weakly compact operator, the product $S \circ T$ is ${}_{au}M$-weakly compact.*

Proof.

(1) ⇒ (2) Let $T: E \longrightarrow Y$ and $S: Y \longrightarrow Z$ be ${}_{au}M$-weakly compact and weakly compact operators, respectively. It follows from Theorem 4, that T' is ${}_{au}L$-weakly compact, and since S' is weakly compact, by Theorem 4, then $(S \circ T)' = T' \circ S'$ is ${}_{au}L$-weakly compact. We conclude from Theorem 4, that $S \circ T$ is ${}_{au}M$-weakly compact.

(2) ⇒ (3) Obvious.

(3) ⇒ (1) Let (x_n) be a disjoint sequence of E and (f_n) a weakly convergent sequence of $B_{Y'}$. Consider the operators $S: Y \longrightarrow c_0$ defined by:

$$\forall y \in Y,\ S(y) = (f_1(y), f_2(y), \ldots),$$
$$\forall (\xi_i) \in c_0,\ f_n((\xi_i)) = \xi_n.$$

Thus $S': l_1 \longrightarrow X'$ satisfies $S'(\alpha_1, \alpha_2, \ldots) = \Sigma \alpha_n f_n$ for all $(\alpha_n) \in l_1$. So by Theorem 5.26 of [1], S' is weakly compact, and consequently the operator S is itself weakly compact. By our hypothesis, the product operator $S \circ T$ is ${}_{au}M$-weakly compact. Thus

$$(|f_n \circ T| \wedge g)(x_n) = (|F_n \circ S \circ T| \wedge g)(x_n) \rightarrow\ 0,$$

as desired.

□

Now, we can establish the following result based on Theorem 8 in Wnuk's paper [13].

Corollary 3. *For a Banach lattice E the following assertions are equivalent*

1. *E' has the positive unbounded Schur property.*
2. *For each Banach space Y every weakly compact operator $T: E \longrightarrow Y$ is ${}_{au}M$-weakly compact.*
3. *Every weakly compact operator $T: E \longrightarrow c_0$ is ${}_{au}M$-weakly compact.*

The following theorem establishes a condition that is sufficient for the equivalence between the classes of ${}_{au}L$-weakly compact operators and positive almost Dunford-Pettis operators, as well as between the classes of ${}_{au}M$-weakly compact operators and semi-compact operators.

Proposition 4. *Let E and F be two nonzero Banach lattices. Then the following statements are equivalent:*

1. *Each positive almost Dunford-Pettis operator $T: E \longrightarrow F$ is ${}_{au}L$-weakly compact.*
2. *The norm of F is order continuous.*

Proof. (2) $\Rightarrow$ (1) Follows from Proposition 2.4 of [4].

(1) $\Rightarrow$ (2) Assume by way of contradiction that the norm of F is not order continuous. We need to construct a positive operator which is almost Dunford-Pettis but not ${}_{au}L$-weakly compact. Since the norm of F is not order continuous, by Theorem 4.14 of [1], there exists a vector $y \in F_+$ and a disjoint sequence $(y_n) \subset [-y, y]$ such that $\|y_n\| \nrightarrow 0$. On the other hand, as E is nonzero, we may fix $u \in E_+$ and pick a $\phi \in (E')_+$ such that $\phi(u) = \|u\| = 1$ holds. Now, we consider operator $T : E \to F$ defined by:

$$T(x) = \phi(x) \,.\, y$$

for each $x \in E$. Obviously, T is a positive operator and is compact (its rank is one). Hence, it is almost Dunford-Pettis. But it is not an ${}_{au}L$-weakly compact. If not, as the singleton u is a weakly compact subset of E, and $T(u) = \phi(u).\, y = y$, the singleton y is an unbounded L-weakly compact subset of F. Since disjoint sequence $(y_n) \subset sol(\{y\})$, we have $\|y_n\| = \||y_n| \wedge u\| \to 0$, which is a contradiction. $\square$

Lemma 4. *For every $(x_n) \subset E_+$ and every sequence (f_n) of X', the following assertions are equivalent*

1. $|f_n \circ T|(x_n) = \sup\{|f_n \circ T(y)| : \; |y| \leq x_n\} \to 0$.
2. *For each sequence (y_n) of E such that $|y_n| \leq x_n$, $f_n \circ T(y_n) \to 0$.*

Theorem 6. *Let E be a Banach lattice and X be a Banach space. If the norm on E' is order continuous, then each Dunford-Pettis operator $T : E \to X$ is ${}_{au}M$-weakly compact.*

Proof. Let $T : E \to X$ be a Dunford-Pettis operator, (x_n) be disjoint sequence of B_E and (f_n) a weakly convergent sequence of $B_{X'}$. Without loss of generality, (x_n) is positive terms. Let (y_n) be sequence of E that $|y_n| \leq x_n$ for all $n \in \mathbb{N}$. Clearly, (y_n) is disjoint sequence of B_E. By Theorem 2.4.14 from [9], then $y_n \xrightarrow{w} 0$. Consequently $\|T(y_n)\| \to 0$. Therefore by Lemma 5, $|f_n \circ T|(x_n) \to 0$. Then

$$(|f_n \circ T| \wedge g)(x_n) \to 0$$

for all $g \in (E')_+$. The proof follows. $\square$

Example 4. If E' is order continuous norm, an ${}_{au}M$-weakly compact operator $T : E \to X$ need not be Dunford-Pettis. For instance, let $E = c_0$. The ${}_{au}M$-weakly compact operator $Id_{c_0} : c_0 \to c_0$ is not Dunford-Pettis.

Theorem 7. *Let E be a Banach lattice and X be a Banach space. If the norm on E and E' are order continuous, then every operator $T : X \longrightarrow E$ is ${}_{au}L$-weakly compact.*

Proof. Let A be relatively weakly compact subset of X and (y_n) be disjoint sequence contained in the solid hull of $T(A)$. Since E' is order continuous norm, by Theorem 2.4.14 from [9], $y_n \xrightarrow{w} 0$. By Proposition 6.3 from [5], disjoint sequence (y_n) is unbounded norm-null. Then T is an ${}_{au}L$-weakly compact operator. $\square$

Corollary 4. *Let X be a Banach space. Every operator $T : X \longrightarrow c_0$ is ${}_{au}L$-weakly compact.*

Theorem 8. *Let X be a nonzero Banach space and E be a Banach lattice. Then the following statements are equivalent:*

(1) *Every semi-compact operator $T : X \longrightarrow E$ is ${}_{au}L$-weakly compact.*
(2) *The norm of E is order continuous.*

Proof.

(2) ⇒ (1) If the norm of E is order continuous, then by Corollary 3.6.4 of [9], semi-compact operator $T : X \longrightarrow E$ is L-weakly compact. It is obvious that every L-weakly compact operator is ${}_{au}L$-weakly compact. Hence T is ${}_{au}L$-weakly compact.

(1) ⇒ (2) Assume by way of contradiction that the norm of E is not order continuous, we need to construct an operator which is semi-compact but not ${}_{au}L$-weakly compact.

Since the norm of E is not order continuous, by Theorem 4.14 of [1], there exists $y \in E_+$ and a disjoint sequence $(y_n) \subset [-y, y]$ such that $\|y_n\| \nrightarrow 0$. On the other hand, since X is nonzero, we can choose a fixed $u \in X$ and select a $\phi \in X^{\prime\prime}$ such that $\phi(u) = |u| = 1$. Now we consider operator $T : X \longrightarrow E$ defined by:

$$T(x) = \phi(x) \, . \, y$$

for each $x \in X$. Clearly, T is not a weakly compact operator, although it is semi-compact due to its compactness (since its rank is one). To see this, note that the singleton u is a weakly compact subset of X, and $T(u) = \phi(u) \cdot y = y$. However, the singleton y is an unbounded L-weakly compact subset of E. This can be seen by considering a disjoint sequence $(y_n) \subset \mathrm{sol}(y)$. We then have $|y_n| = |y_n \wedge y| \to 0$, which leads to a contradiction.

□

Theorem 9. *Let E and F be two nonzero Banach lattices. Then the following statements are equivalent:*

(1) *Every positive semi-compact operator $T : E \longrightarrow F$ is ${}_{au}M$-weakly compact.*
(2) *The norm of E' is order continuous.*

Proof.

(2) ⇒ (1) According to Corollary 3.3 of [2], it can be established that if a positive operator $T : E \longrightarrow F$ is semi-compact, then its adjoint $T' : F' \longrightarrow E'$ is an almost Dunford-Pettis operator. Additionally, since the norm of E' is order continuous (as Proposition 2.4 of [4]), T' can be characterized as an almost L-weakly compact operator. Utilizing Theorem 2.5(1) from [4], it follows that T is ${}_{au}M$-weakly compact.

(1) $\Rightarrow$ (2) Assume by way of contradiction that the norm of E' is not order continuous, we need to construct a positive operator which is semi-compact, but not ${}_{au}M$-weakly compact. Since the norm of E' is not order continuous, by Theorem 4.14 of [1], there exists a vector $\phi \in (E')_+$ and a disjoint sequence $(\phi_n) \subset [-\phi, \phi]$ such that $\|\phi_n\| \nrightarrow 0$. On the other hand, as F is nonzero, we may fix $y \in F_+$ and pick a vector $g \in (F')_+$ such that $g(y) = \|y\| = 1$ holds. Now, we consider operator $T : E \longrightarrow F$ defined by

$$T(x) = \phi(x)\,.y$$

for each $x \in E$. Obviously, T is positive and semi-compact operator as it is compact (its rank is one). But it is not an ${}_{au}M$-weakly compact operator. In fact, by Theorem 4, we only need to show that its adjoint $T' : F' \longrightarrow E'$ defined by $T'(f) = f(y)\,.\,\phi$ for each $f \in F'$, is not ${}_{au}L$-weakly compact. If not, as the singleton $\{g\}$ is weakly compact subset of X', and $T'(g) = g(y).\,\phi = \phi$, the singleton $\{\phi\}$ is unbounded L-weakly compact subset of E'. Since disjoint sequence $(\phi_n) \subset sol(\{\phi\})$, we have $\|\phi_n\| = \|\phi_n \wedge \phi\| \to 0$, which is contradiction.

□

Note: In the following statements, we assume that the operator T is bounded.

Remark 2. If T from E into F is an order bounded operator then by Theorem 1.23 in [1], T' from F' into E' is order bounded and for each $0 \le f \in F'$ and each $x \in E_+$ we have

$$\langle f, |Tx|\rangle \le \langle |T'|f, x\rangle\,.$$

Also, if the modulus of T exists, it follows from $\pm T \le |T|$ that $\pm T' \le |T|'$. That is, $|T'| \le |T|'$ holds whenever the modulus of T exists.

Remark 3. Let T be an order bounded operator from E into F, where F is Archimedean and preserves disjointness. It follows that

$$|T|' = |T'|.$$

Proposition 5. *Let S, T from E into F be two positive operators such that $0 \le S \le T$ and $T \in LW_{au}(E, F)$. Then by one of the following conditions, we have $S \in LW_{au}(E, F)$.*

1. *T is an order bounded operator with F Archimedean preserves disjointness.*
2. *E has weakly sequentially continuous lattice operations.*
3. *E is AM- space.*
4. *E has the Schur property.*

Proof. Let T be an ${}_{au}L$-weakly compact operator and $(x_n) \subseteq E$ be weakly convergent sequence and (y_i) be an order bounded disjoint sequence contained in the solid hull of $\{S(x_n) : n \in \mathbb{N}\}$. It follows that there exists a subsequence

(x_{n_i}) of (x_n) such that $|y_i| \leq |Sx_{n_i}|$. Based on condition (1) and Theorem 2.40 from [1], we can conclude the following:

$$|y_i| \leq |Sx_{n_i}| \leq S|x_{n_i}| \leq T|x_{n_i}| = |Tx_{n_i}|,$$

for all $i \in \mathbb{N}$. Since T is ${}_{au}L$-weakly compact, hence by Theorem 2, $y_i \to 0$. Also by Theorem 2, $S \in LW_{au}(E, F)$.

By one of the conditions (2), (3), and (4) we conclude that $(|x_n|)$ is weakly convergent sequence. We have

$$|y_i| \leq |Sx_{n_i}| \leq S|x_{n_i}| \leq T|x_{n_i}|,$$

for all $i \in \mathbb{N}$. Since T is ${}_{au}L$-weakly compact, hence by Theorem 2, $y_i \to 0$. Thus by Theorem 2, $S \in LW_{au}(E, F)$. □

Based on Theorem 4 and Proposition 5, we can derive the following proposition.

Proposition 6. *Let S, T from E into F be two positive operators such that $0 \leq S \leq T$ and $T \in MW_{au}(E, F)$. Then by one of the following conditions, we have $S \in MW_{au}(E, F)$.*

(1) *T' is an order bounded operator with F' Archimedean preserves disjointness.*
(2) *F' has weakly sequentially continuous lattice operations.*
(3) *F' is AM- space.*
(4) *F' has the Schur property.*

Theorem 10. *If T is an order bounded operator mapping from E to F, where F is an Archimedean lattice that preserves disjointness, then the following statements are true.*

(1) *If $T \in LW_{au}(E, F)$, then $|T| \in LW_{au}(E, F)$.*
(2) *If T' be disjointness preservess and $T \in MW_{au}(E, F)$, then $|T| \in MW_{au}(E, F)$.*

Proof. (1) Let $A \subset E$ be relatively weakly compact and (y_n) be an order bounded disjoint sequence contained in the solid hull of $|T|(A)$. Then, there exists a sequence (x_n) of A such that $|y_n| \leq ||T|x_n|$. By Theorem 2.40 from [1] we have

$$|y_n| \leq ||T|x_n| \leq |T||x_n| = |Tx_n|,$$

for all $n \in \mathbb{N}$. Since T is ${}_{au}L$-weakly compact, hence by Theorem 2, $y_n \to 0$.

By Theorem 2, we have $|T| \in LW_{au}(E, F)$.

(2) By apply (1) and Theorem 10, proof follows.

□

Theorem 11. *By one of the following conditions, if $|T| \in LW_{au}(E, F)$ then $T \in LW_{au}(E, F)$.*

(1) *T is an order bounded operator and F Archimedean with preserves disjointness.*
(2) *E is AM- space.*

(3) *E has the Schur property.*
(4) *E has weakly sequentially continuous lattice operations.*

Proof. Let $(x_n) \subset E$ be weakly convergent sequence and (y_i) be an order bounded disjoint sequence contained in the solid hull of $\{T(x_n) : n \in \mathbb{N}\}$. So, there exists a subsequence (x_{n_i}) of (x_n) such that $|y_i| \leq |Tx_{n_i}|$. Then, by condition (1) and Theorem 2.40 from [1], we have

$$|y_i| \leq |Tx_{n_i}| = |T||x_{n_i}|,$$

for all $i \in \mathbb{N}$. Since $|T|$ is ${}_{au}L$-weakly compact, hence by Theorem 2, $y_i \to 0$. Thus, by Theorem 2, $T \in LW_{au}(E, F)$.

Now, based on one of the conditions (2), (3), and (4), we can conclude that $(||x_n||)$ is a weakly convergent sequence. We have

$$|y_i| \leq |Tx_{n_i}| \leq |T||x_{n_i}|,$$

for all $n \in \mathbb{N}$. Since $|T|$ is ${}_{au}L$-weakly compact, hence by Theorem 2, $y_i \to 0$ and proof follows. □

Theorem 12. *By one of the following conditions, if $|T| \in MW_{au}(E, F)$ then $T \in MW_{au}(E, F)$.*

(1) *T' is an order bounded operator disjointness preserves.*
(2) *F' is AM- space.*
(3) *F' has the Schur property.*
(4) *F' has weakly sequentially continuous lattice operations.*

Proof. Let $|T|$ be an ${}_{au}M$-weakly compact. Then by Theorem 4, $|T|'$ is ${}_{au}L$-weakly compact. Since $|T'| \leq |T|'$, by Proposition 5, $|T'|$ is ${}_{au}L$-weakly compact. We conclude from Theorem 12 that T' is ${}_{au}L$-weakly compact. Hence by Theorem 4, T is ${}_{au}M$-weakly compact. □

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