Manipulating multiqudit entanglement witnesses by using linear programming

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A class of entanglement witnesses (EWs) called reduction-type entanglement witnesses is introduced, which can detect some multipartite entangled states including positive partial transpose ones with Hilbert space of dimension $d_1 \otimes d_2 \otimes \cdots \otimes d_n$. In fact the feasible regions of these EWs turn out to be convex polygons and hence the manipulation of them reduces to linear programming which can be solved exactly by using the simplex method. The decomposability and nondecomposability of these EWs are studied and it is shown that it has a close connection with eigenvalues and optimality of EWs. Also using the Jamiołkowski isomorphism, the corresponding possible positive maps, including the generalized reduction maps of Hall [Phys. Rev. A **72**, 022311 (2005)] are obtained.

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I. INTRODUCTION

In the past decade quantum entanglement has attracted much attention in connection with theory of quantum information and computation. This is because of the potential resource that entanglement provides for quantum communication and information processing [1–3]. How to characterize and measure the entanglement is a basic problem. Although, in the case of pure states of bipartite systems it is to check whether a given state is, or is not entangled, the question is yet an open problem in the case of mixed states [4–7].

Among the known criteria for characterizing entanglement, entanglement witness (EW) [8,9] is an important one for detecting the presence of entanglement. The EWs are nonpositive Hermitian operators which can detect the presence of entanglement. Unlike the positive partial transpose (PPT) criterion [9,10], which is a necessary and sufficient condition for determining entangled states living in Hilbert spaces $\mathcal{H}_2 \otimes \mathcal{H}_2$ and $\mathcal{H}_2 \otimes \mathcal{H}_3$, ($\mathcal{H}_d$ denotes the Hilbert space with dimension d), for higher dimensions, this criterion is a necessary one, the EW criterion is still sufficient and a necessary one regardless of the dimension of quantum systems, since for every entangled state there exists an EW detecting it.

The most important characteristic of the separability problem is the fact that the separable states form a convex set and the existence of EWs is a direct consequence of this convexity. Convex optimization, and in particular semidefinite programming [11], have proven useful for quantum optimization problems such as a test for distinguishing an entangled from a separable quantum state [12–17] and the optimization of the structure of a continuous measurement for a linear quantum system [18]. Convexity also plays a central role in our work which provides a classical design of constructing EWs with certain properties.

In this paper, by using the prescription of Refs. [19,20], i.e., reducing the manipulation of EWs to linear programming (LP) which is a special case of convex optimization [21], we find a class of EWs called reduction-type entanglement witnesses (REWs) for multipartite systems with arbitrary dimensions. However, unlike the previous works [19,20] in which determining feasible regions needs to use numerical calculations, here, steps towards the finding REWs use exact LP method without appealing to any numerical or approximate calculations. It is interesting that our construction will allow us to characterize the extreme points of a feasible region. Indeed the feasible region for this optimization problem forms a polygon by itself, hence the problem can be cast as LP that optimizes a linear function of positive variables subject to linear constraints, and the simplex method is the easiest way of solving it.

The REWs also have the property that they can be written in terms of some positive operators and optimal EWs corresponding to hyperplanes surrounding the feasible region. Moreover, in most cases the decomposability of the REWs is rather determined, where just two eigenvalues of them play an important role in this issue. Another consequence of such EWs is positive maps [22] including the generalized reduction map [23], which can be obtained from REWs or their tensor products, via Jamiołkowski isomorphism [24].

The paper is organized as follows: In Sec. II we give a brief review of EWs. In Sec. III we explain the general scheme of LP. In Sec. IV, we introduce a class of EWs which can be put in the realm of LP, since their feasible regions are convex polygons (indeed simplex) which can be exactly determined. In Sec. V, we show that all EWs corresponding to hyperplanes surrounding the feasible regions are optimal. Section VI is devoted to some interesting examples such as: multiqubit REWs, $(d \otimes d \cdots \otimes d)$ multiqudit REWs, and $2 \otimes 3 \otimes 4$ REW. Section VII deals with two examples of LP type, where the first one can be solved exactly by the prescription of this paper. In Sec. VIII we introduce some entangled and PPT states which can be detected by REWs and the decomposability of REWs is discussed. In Sec. IX by using Jamiołkowski isomorphism, the relation between

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REWs and positive maps is explained. The paper is ended with a brief conclusion and three appendixes.

II. ENTANGLEMENT WITNESS

As mentioned in the introduction, one of the pragmatic approach to detect entanglement is to construct entanglement witnesses. First we recall the definitions of separability and entanglement. Following Refs. [25,26], these definitions can be extended in the following natural manner. Consider a system shared by n parties $\{M_i\}_{i=1}^n$. We call a k-partite split a partition of the system into $k\leqslant n$ set $\{S_i\}_{i=1}^k$, where each may be composed of several original parties. A given state $\rho \in \mathcal{B}(\mathcal{H}_{d_1}\otimes\cdots\otimes\mathcal{H}_{d_k})$, associated with some k-partite split, is a m-separable state if it is possible to find a convex decomposition for it such that in each pure state term at most m parties are entangled among each other, but not with any member of the other group of $n-m$ parties. For example, every 1-separable density operator $\rho\in\mathcal{B}(\mathcal{H})$ (the Hilbert space of bounded operators acting on $\mathcal{H}=\mathcal{H}_{d_1}\otimes\cdots\otimes\mathcal{H}_{d_n}$) is fully separable which can be written as

$$\rho_s=\sum_i p_i|\alpha_i^{(1)}\rangle\langle\alpha_i^{(1)}|\otimes|\alpha_i^{(2)}\rangle\langle\alpha_i^{(2)}|\otimes\cdots\otimes|\alpha_i^{(n)}\rangle\langle\alpha_i^{(n)}| \tag{1}$$

with $p_i\geqslant 0$ and $\Sigma_i p_i=1$, hence the set of all fully separable states (hereafter, the separable states mean the fully separable states) is a convex set called the convex set of separable states (CSSS).

Definition 1. A Hermitian operator $\mathcal{W}$ is called an entanglement witness detecting the entangled state ρ_e if $\mathrm{Tr}(\mathcal{W}\rho_e)<0$ and $\mathrm{Tr}(\mathcal{W}\rho_s)\geqslant 0$ for all separable states ρ_s.

So, if we have a density operator ρ and we measure $\mathrm{Tr}(\mathcal{W}\rho)<0$, we can be sure that ρ is entangled. This definition has a clear geometrical meaning. The expectation value of an observable depends linearly on the state. Thus, the set of states where $\mathrm{Tr}(\mathcal{W}\rho)=0$ holds is a hyperplane in the set of all states, cutting this set into two parts. In the part with $\mathrm{Tr}(\mathcal{W}\rho)>0$ lies the set of all separable states, the other part [with $\mathrm{Tr}(\mathcal{W}\rho)<0$] is the set of state detected by $\mathcal{W}$. From this geometrical interpretation it follows that all entangled states can be detected by witness. Indeed for each entangled state ρ_e there exists an entanglement witness detecting it [9].

Usually one is interested in a selected group of witnesses operators called decomposable EWs which based on the notion of partial transposition. Let $\mathcal{Q}\in\mathcal{B}(\mathcal{H}_{d_1}\otimes\cdots\otimes\mathcal{H}_{d_n})$ such that

$$\mathcal{Q}=\sum_{i_1,j_1=0}^{d_1-1}\cdots\sum_{i_n,j_n=0}^{d_n-1} q_{i_1,\ldots,i_n,j_1,\ldots,j_n}\bigotimes_{k=1}^{n}|i_k\rangle\langle j_k|, \tag{2}$$

where $|i_r\rangle$ forms an orthonormal basis for $\mathcal{H}_{d_r}$. The partial transpose of $\mathcal{Q}$ with respect to any subset $S\subseteq\mathcal{N}$ defined as

$$\mathcal{Q}^{T_S}=\sum_{i_1,j_i=0}^{d_1-1}\cdots\sum_{i_n,j_n=0}^{d_n-1} q_{i_1,\ldots,i_n,j_1,\ldots,j_n}\bigotimes_{k=1}^{n}O_{i_kj_k},$$

such that

$$O_{i_kj_k}=\begin{cases}|j_k\rangle\langle i_k| & \text{if } k\in S,\\ |i_k\rangle\langle j_k| & \text{if } k\notin S.\end{cases} \tag{3}$$

Definition 2. An EW $\mathcal{W}$ is decomposable EW (d-EW) iff there exists operators $\mathcal{P}$, $\mathcal{Q}_i$ $(i=1,\ldots,m)$ with

$$\mathcal{W}=\mathcal{P}+\mathcal{Q}_1^{T_A}+\mathcal{Q}_2^{T_B}+\cdots+\mathcal{Q}_m^{T_Z},\quad \mathcal{P},\mathcal{Q}_i\geqslant 0,\quad i=1,\ldots,m, \tag{4}$$

where $A,B,\ldots$ are nonempty subsystems of $\mathcal{N}$ and the upper index T_S denotes partial transpose with respect to subsystem $S\subset\mathcal{N}$. An EW $\mathcal{W}$ is nondecomposable EW if it cannot be set in the form of Eq. (4) (see Ref. [17]).

One should notice that only nondecomposable EWs can detect PPT entangled states. ρ is called PPT state when

$$\rho^{T_S}\geqslant 0,\qquad \forall\, S\subseteq\mathcal{N} \tag{5}$$

that is, those density operators which have positive partial transpose with respect to each subsystem [27].

III. MANIPULATING EWS BY LP METHOD

This section deals with the basic definition of LP and a general scheme to construct EWs by an exact LP method.

To this aim first we consider a Hermitian operator $\mathcal{W}$ with some negative eigenvalues

$$\mathcal{W}=\sum_i a_iQ_i, \tag{6}$$

where Q_i are positive operators with $0\leqslant\mathrm{Tr}(Q_i\rho_s)\leqslant 1$ for every separable states ρ_s and $a_i\in\mathbb{R}$ are the parameters whose ranges must be determined such that $\mathcal{W}$ be an EW. Note that, the condition $0\leqslant\mathrm{Tr}(Q_i\rho_s)\leqslant 1$ is not always required. It is used here only to simplify analyzing the problem and pave the way to generalize the prescription to multiqudits with arbitrary higher dimensions as will be discussed in the following.

As ρ_s varies over CSSS, the map $P_i=\mathrm{Tr}(Q_i\rho_s)$ maps CSSS into a convex region called the feasible region (inside the hypercube defined by $0\leqslant P_i\leqslant 1$). Now, we try to choose the real parameters a_i such that the operator $\mathcal{W}$ given in (6) possesses at least one negative eigenvalue and its expectation value over any separable state be nonnegative, i.e., the condition $\mathrm{Tr}(\mathcal{W}\rho_s)=\Sigma_i a_iP_i\geqslant 0$ must be satisfied for all P_i belonging to the feasible region.

Therefore, for determination of EWs of type (6), one needs to determine the minimum value of $\Sigma_i a_iP_i$ over the feasible region and hence the problem reduces to the optimization of the linear function $\Sigma_i a_iP_i$ over the convex set of the feasible region.

We note that the minimum value of $\mathcal{F}_{\mathcal{W}}:=\mathrm{Tr}(\mathcal{W}\rho_s)$ is achieved for the pure product state, since every mixed separable state ρ_s can be written as a convex combination of pure product states (due to the convexity of the separable region), i.e., $\rho_s=\Sigma_i p_i|\gamma_i\rangle\langle\gamma_i|$ with $|\gamma_i\rangle=|\alpha_i^{(1)}\rangle|\alpha_i^{(2)}\rangle\cdots|\alpha_i^{(n)}\rangle$ and $p_i\geqslant 0$, $\Sigma_i p_i=1$, hence we have

$$\mathrm{Tr}(\mathcal{W}\rho_s) = \sum_i p_i\, \mathrm{Tr}(\mathcal{W}|\gamma_i\rangle\langle\gamma_i|) \geq C_{\min} \tag{7}$$

with $C_{\min} := \min \mathcal{F}_{\mathcal{W}} = \min_{|\gamma\rangle \in D_{\mathrm{prod}}} \mathrm{Tr}(\mathcal{W}|\gamma\rangle\langle\gamma|)$, where D_{prod} denotes the set of product states. Thus we need to find the pure product state $|\gamma_{\min}\rangle$ which minimize $\mathrm{Tr}(\mathcal{W}|\gamma\rangle\langle\gamma|)$. For the cases that the feasible regions are simplices (or at most convex polygons), manipulating these EWs amounts to

$$\text{minimize } \mathcal{F}_{\mathcal{W}} = \sum_i a_i P_i$$

$$\text{subject to } \sum_i (c_{ij}P_i - d_j) \geq 0, \quad j = 1,2, \ldots , \tag{8}$$

where c_{ij}, d_i $i,j = 1,2, \ldots .$ are parameters of hyperplanes surrounding the feasible regions.

One can calculate the distributions P_i, consistent with the aforementioned optimization problem, from the information about the boundary of the feasible region. To achieve the feasible region we obtain the extreme points corresponding to the product distributions P_i for every given product state by applying the special conditions on the parameters a_i. In fact, $\mathcal{F}_{\mathcal{W}}$ themselves are functions of the product distributions, and they are in turn functions of $|\gamma\rangle$. They are not real variables of $|\gamma\rangle$ but the product states will be multiplicative. If this feasible region constructs a polygon by itself, the corresponding boundary points of the convex hull will minimize exactly $\mathcal{F}_{\mathcal{W}}$ in Eq. (8). This problem is called exact LP and the simplex method is the easiest way of solving it [21].

If the feasible region is not a polygon, with the help of tangent planes in this region at points which are determined either analytically or numerically, one can define a new convex hull which is a polygon encircling the feasible region. The points on the boundary of the polygon can approximately determine the minimum value of $\mathcal{F}_{\mathcal{W}}$ in (8). Thus the approximated value is obtained via LP.

Here in this paper we are interested in the EWs where the feasible regions are convex polygons and hence the problem can be solved by LP and the simplex method exactly.

IV. $(d_1 \otimes d_2 \otimes \cdots \otimes d_n)$ MULTIQUDIT REDUCTION-TYPE EWS

In this section we consider n particles with arbitrary dimensions. Without loss of generality particles can be arranged so that $d_1 \leq d_2 \leq \cdots \leq d_n$. The discussion of some special cases is postponed to Sec. VI. We introduce and parametrize the multiqudit reduction-type entanglement witnesses labeled by subscript R as

$$\mathcal{W}_R^{(n)} = \sum_{S \subsetneqq N'} b_S \sigma_S + d_1 b_{2,\ldots,n} |\psi_{00\cdots 0}\rangle\langle\psi_{00\cdots 0}| + \sum_{S \subsetneqq N'} a'_S \sigma'_S, \tag{9}$$

where $N' = \{2, \ldots, n\}$, with $b_{\emptyset} = b_1$, $a'_{\emptyset} = a'_1$ and σ_S, σ'_S defined as

$$\sigma_S = \sum_{i=0}^{d_1-1} |i\rangle\langle i| \otimes O_i^{(2)} \otimes \cdots \otimes O_i^{(n)} \quad \text{with } O_i^{(k)} = \begin{cases} |i\rangle\langle i| & \text{if } k \in S, \\ I & \text{if } k \notin S, \end{cases} \tag{10}$$

and

$$\sigma'_S = \sum_{i_1=0}^{d_1-1} \sum_{i_2=0}^{d_2-1} \cdots \sum_{i_n=0}^{d_n-1} |i_1\rangle\langle i_1| \otimes O'^{(2)}_{i_2} \otimes \cdots \otimes O'^{(n)}_{i_n} \tag{11}$$

with

$$O'^{(k)}_{i_k} = \begin{cases} |i_1\rangle\langle i_1| \delta_{i_1 i_k} & \text{if } k \in S, \\ |i_k\rangle\langle i_k| \prod_{k \neq m=1}^{n} (1 - \delta_{i_k i_m}) & \text{if } k \notin S, \quad |S| \leq n-3, \text{ and } i_2, \ldots, i_n \leq d_1 - 1, \\ 0 & \text{if } k \notin S, \quad |S| = n-2, \text{ and } i_2, \ldots, i_n \leq d_1 - 1, \\ |i_k\rangle\langle i_k| (1 - \delta_{i_k i_1}) & \text{otherwise,} \end{cases}$$

respectively, and

$$|\psi_{00\cdots 0}\rangle := \frac{1}{\sqrt{d_1}} \sum_{i=0}^{d_1-1} |i^{(1)}\rangle|i^{(2)}\rangle \cdots |i^{(n)}\rangle \tag{12}$$

is a maximally entangled state and all coefficients $b_S, b_{2,\ldots,n}, a'_S$ are the real parameters whose ranges must be determined such that $\mathcal{W}_R^{(n)}$ be an EW. In this notation we have $\sigma'_{\emptyset} = \sigma'_1$. Obviously for the multiqubit system none of σ'_S exists. The number of P'_S depends on the dimensions of particles d_i's and it can take one of the following values:

$$m=\begin{cases}\sum_{|S|=n-d}^{n-3}\mathbf{C}_{|S|}^{n-1} & \text{if } d_1=d_2=\cdots=d_n=d,\\ \sum_{|S|=0}^{n-2}\mathbf{C}_{|S|}^{n-1}-(2^{m-1}-1) & \text{if } d_1=\cdots=d_m<d_{m+1}\leqslant\cdots\leqslant d_n,\end{cases}$$

where $\mathbf{C}_m^n=\frac{n!}{m!(n-m)!}$. We introduce the new more convenient parameters $a_1=b_1=b_\varnothing$, $a_S=b_S+\Sigma_{S'\subsetneqq S}b_{S'}$, instead of b_S's. In order to turn the observable (9) to an EW, we need to choose its parameters in such a way that it becomes a nonpositive operator with positive expectation values in any pure product state $|\gamma\rangle=|\alpha^{(1)}\rangle\otimes\cdots\otimes|\alpha^{(n)}\rangle$.

Now it is time to reduce the problem to the LP one. Io order to determine the feasible region, we need to know the apexes, namely the extremum points, to construct the hyperplanes surrounding the feasible regions. Suppose that $|\gamma\rangle$ be a pure product state with $|\alpha^{(k)}\rangle=(\alpha_0^{(k)},\alpha_1^{(k)},\ldots,\alpha_{d_k-1}^{(k)})^T$ where $k=1,\ldots,n$ and d_k denotes the dimension of $|\alpha^{(k)}\rangle$. Let

$$P_S\coloneqq \mathrm{Tr}(\sigma_S|\gamma\rangle\langle\gamma|)=\sum_{i=0}^{d_1-1}|\beta_i^{(1)}\beta_i^{(2)}\cdots\beta_i^{(n)}|^2,$$

$$\beta_i^{(k)}=\begin{cases}\alpha_i^{(k)} & \text{if } k\in\{1\}\cup S,\\ 1 & \text{if } k\notin\{1\}\cup S,\end{cases}$$

$$P_{2,\ldots,n}\coloneqq d_1|\langle\psi_{00\ldots0}|\gamma\rangle|^2=\left|\sum_{i=0}^{d_1-1}\alpha_i^{(1)}\alpha_i^{(2)}\cdots\alpha_i^{(n)}\right|^2,$$

$$P'_S\coloneqq \mathrm{Tr}(\sigma'_S|\gamma\rangle\langle\gamma|),$$

which all lie in the interval [0,1] (see Appendix A). The number of P_S is $2^{n-1}-1$ (cardinality of the power set of N' excepted $\{\varnothing\}$).

The extremum points or apexes consist of the following.

(a) Origin, $P_S=0$, $P'_S=0$ $\forall S\subseteqq N'$ which corresponds to the following choice of pure product state:

$$|\alpha^{(1)}\rangle=(1\ 0\ 0\ \cdots\ 0)^T,$$

$$|\alpha^{(k)}\rangle=(0\ 1\ 0\ \cdots\ 0)^T,\quad k\in N'.$$

(b) $P_S=1$, $P_{S'}=1$ $\forall S'\subseteqq S$, $P_{N'}=0$, $P_{S''}=0$ $\forall S''\subseteqq N'\backslash S$, $P'_S=0$ $\forall S\subseteqq N'$ can be reached by choosing the following pure product states:

$$|\alpha^{(k)}\rangle=(1\ 0\ 0\ \cdots\ 0)^T,\quad k\in\{1\}\cup S,$$

$$|\alpha^{(k)}\rangle=(0\ 1\ 0\ \cdots\ 0)^T,\quad k\notin\{1\}\cup S.$$

Obviously if $P_S=1$, then for all $S'\subseteqq S$, $P_{S'}=1$, thus for $S=N'$, we get the following important apex.

(c) $P_S=1$, $PS'=0$ $\forall S\subseteqq N'$, which can be obtained by choosing

$$|\alpha^{(k)}\rangle=(1\ 0\ 0\cdots0)^T\ \ \forall\ k\in\{1\}\cup N'.$$

(d) $P'_S=1$, $P_{S'}=1$ $\forall S'\subseteqq S\subseteqq N'$, the others are zero.

The last category arises from pure product state with the components of the form $\alpha_0^{(1)}=\alpha_0^{(k)}=1$, if $k\in S$ and $\alpha_{k-1}^{(k)}=1$, if $k\notin S$, i.e.,

$$|\alpha^{(k)}\rangle=(1\ 0\ 0\cdots0)^T,\quad k\in\{1\}\cup S,$$

$$|\alpha^{(k)}\rangle=(0\ 0\cdots0\ \underbrace{1}_{k-1\text{th}}\ \ 0\cdots0)^T,\ k\notin\{1\}\cup S.$$

Regarding the above consideration, we are now ready to state the feasible region.

To this aim we first prove that, $2^{n-1}+m$ extremum points obtained above form the apexes of $(2^{n-1}+m)$-simplex in Euclidean space of dimension $N=2^{n-1}+m-1$. For this purpose we consider the convex hull of theses points, i.e., draw $N+1$ hyperplanes passing through each combination of N points out of the $N+1$ ones ($\mathbf{C}_N^{N+1}=N+1$). Now, we get a bounded region formed from their intersection which is the required feasible region of $\mathcal{W}_R^{(n)}$ and it is obviously a $(2^{n-1}+m)$ simplex. It is straight forward to show that the feasible region can be obtained by taking the expectation values of

$$^{(S)}\mathcal{W}_{\mathrm{opt}}^{(n)}=a_S\bigg(\sigma_S+\sum_{S\subsetneqq S'\neq N'}(-1)^{|S|+|S'|}\sigma_{S'}+d_1(-1)^{|S|+|N'|}\times|\psi_{00\cdots0}\rangle\langle\psi_{00\ldots0}|-\sigma'_S\bigg),\quad S\subsetneqq N' \tag{13}$$

in pure product states (see Appendix B). Now, in order for $\mathcal{W}_R^{(n)}$ to be an EW, the following expectation values,

$$\mathcal{F}_{\mathcal{W}}(P_2,P_3,\ldots,P'_1,P'_2)\coloneqq \mathrm{Tr}(\mathcal{W}_R^{(n)}\rho_s)=\sum_{S\subsetneqq N'}b_SP_S+b_{2,\ldots,n}P_{2,\ldots,n}+\sum_{S\subsetneqq N'}a'_SP'_S \tag{14}$$

must be positive. So our task is to solve the following LP problem:

$$\text{minimize}\quad \sum_{S\subsetneqq N'}b_SP_S+b_{2,\ldots,n}P_{2,\ldots,n}+\sum_{S\subsetneqq N'}a'_SP'_S$$

$$\text{subject to}\quad \begin{cases}P_S-P'_S+\sum_{S\subsetneqq S'}(-1)^{|S|+|S'|}P_{S'}\geqslant 0,\\ \forall\ P'_S\geqslant 0.\end{cases} \tag{15}$$

Setting the coordinates of apexes in Eq. (14) yields all $a_S\geqslant 0$ and $a_S+a'_S\geqslant 0$. As we stated in preceding section, all P_S

and P'_S lie in the closed interval [0,1]. Now, $\mathcal{F}_{\mathcal{W}}$ is a linear function of P_S and P'_S and if we require it to be positive on the apexes (which are extremum points), then it will be positive in the whole feasible region.

At the end we need to know all eigenvalues of $\mathcal{W}_R^{(n)}$ which consist of a_S, $a_S+a'_S, \omega_1=a_{N'}-b_{N'}$, and $\omega_2=d_1a_{N'}-(d_1-1)\omega_1$. Since a_S, $a_S+a'_S\geqslant 0$, then one of the remaining eigenvalues ω_1 and ω_2 must be negative to guarantee $\mathcal{W}_R^{(n)}$ to be an EW.

V. OPTIMALITY OF ${}^{(S)}\mathcal{W}_{\text{opt}}^{(n)}$

After determining the feasible regions, one needs to know whether the boundary of EWs is formed by optimal EWs. An EW is optimal if for all positive operators P subtracted from that, it will be no longer an EW [8].

Note that the EWs corresponding to hyperplanes surrounding feasible regions of $\mathcal{W}_R^{(n)}$ are optimal since they cover the simplex feasible region in an optimal way (see Appendix C). Thus, the structure of the optimal EWs ${}^{(S)}\mathcal{W}_{\text{opt}}^{(n)}$ characterizes the boundary of REWs $\mathcal{W}_R^{(n)}$. In fact, from the results of this section it will become clear that we can restrict ourselves to the structure of the optimal EWs corresponding to hyperplanes surrounding feasible regions. In other words, optimal EWs ${}^{(S)}\mathcal{W}_{\text{opt}}^{(n)}$ are tangent to the boundary between separable and nonseparable states.

Another advantage of ${}^{(S)}\mathcal{W}_{\text{opt}}^{(n)}$ is that one can rewrite the $\mathcal{W}_R^{(n)}$ in terms of positive operators $\sigma'_S, |\psi_{00\cdots0}\rangle\langle\psi_{00\cdots0}|$ and some optimal EWs, i.e.,

$$\mathcal{W}_R^{(n)} = \sum_S a_S {}^{(S)}\mathcal{W}_{\text{opt}}^{(n)} + d_1\omega_2|\psi_{00\cdots0}\rangle\langle\psi_{00\cdots0}| + \sum_S (a_S+a'_S)\sigma'_S, \quad S \subsetneq N'. \tag{16}$$

Therefore for positive ω_2, the REWs can be decomposed as

$$\mathcal{W}_R^{(n)} = \sum_S {}^{(S)}\mathcal{Q}^{T(N'\setminus S)} + \mathcal{P}, \tag{17}$$

where ${}^{(S)}\mathcal{Q} := {}^{(S)}\mathcal{W}_{\text{opt}}^{(n)T(N'\setminus S)}$ and $\mathcal{P}$ is a positive operator and in this case $\mathcal{W}_R^{(n)}$ cannot detect PPT entangled states (nonseparable density matrices with positive partial transpose with respect to all particles).

VI. SOME SPECIAL CASES OF $\mathcal{W}_R^{(n)}$

In this section we discuss some special cases of REWs to enlighten the subject.

A. Multiqubit reduction-type EWs

It is important both theoretically and experimentally to study multiqubit entanglement and to provide EWs to verify that in a given multiqubit state, entanglement is really present. Equation (9) for a system of n-qubits reduces to

$$\mathcal{W}_R^{(n)} = \sum_{S\subsetneq N'} b_S\sigma_S + 2b_{2,\ldots,n}|\psi_{00\cdots0}\rangle\langle\psi_{00\cdots0}|. \tag{18}$$

As mentioned before, the dimension of qubit systems does not allow presence of σ'_S.

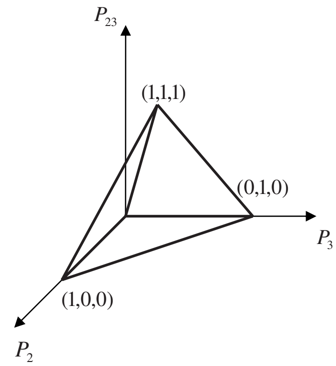

FIG. 1. 3-simplex displaying the feasible region of three-qubit REW.

The number of P's is $2^{n-1}-1$, whereas the number of apexes is 2^{n-1}. Setting the coordinates of apexes in Eq. (14) again indicates that all $a_S\geqslant 0$. The feasible region is 2^{n-1} simplex of dimension $2^{n-1}-1$ surrounded by hypersurfaces defined by (15) with all P''s eliminated.

1. Three qubit $(n=3)$

The first nontrivial example of REWs for a multiqubit system is three-qubit REW,

$$\mathcal{W}_R^{(3)} = a_1I_8 + (a_2-a_1)\sigma_2 + (a_3-a_1)\sigma_3 + 2(a_{2,3}+a_1-a_2-a_3) \times|\psi_{000}\rangle\langle\psi_{000}| \tag{19}$$

with eigenvalues $a_1,a_2,a_3\geqslant 0, \omega_1=a_2+a_3-a_1$, and $\omega_2=2a_{2,3}-\omega_1$. In order to obtain the feasible region, we need the coordinates of apexes which are

$$|\gamma\rangle = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} \to (P_2=0, P_3=0, P_{2,3}=0),$$

$$|\gamma\rangle = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} \to (P_2=1, P_3=0, P_{2,3}=0),$$

$$|\gamma\rangle = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} \to (P_2=0, P_3=1, P_{2,3}=0),$$

$$|\gamma\rangle = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} \to (P_2=1, P_3=1, P_{2,3}=1).$$

As we see the number of P_S is three, while the number of optimal points is four. Therefore, there are four hyperplanes surrounding the feasible region, where the hyperplanes pass through each combination of three points out of four (see Fig. 1). Thus the problem can be reduced to

$$\text{minimize} \quad \text{Tr}(\mathcal{W}_R^{(3)}|\gamma\rangle\langle\gamma|)$$

$$\text{subject to} \quad \begin{cases} 1 + P_{2,3} - P_2 - P_3 \geq 0, \\ P_2 - P_{2,3} \geq 0, \\ P_3 - P_{2,3} \geq 0, \\ P_{2,3} \geq 0, \end{cases} \tag{20}$$

where the above given inequalities follow rather easily by taking the expectation value of the following optimal EWs:

$$^2\mathcal{W}_{\text{opt}}^{(3)} = a_2(\sigma_2 - 2|\psi_{000}\rangle\langle\psi_{000}|),$$

$$^3\mathcal{W}_{\text{opt}}^{(3)} = a_3(\sigma_3 - 2|\psi_{000}\rangle\langle\psi_{000}|),$$

$$^1\mathcal{W}_{\text{opt}}^{(3)} = a_1(I_8 - \sigma_2 - \sigma_3 + 2|\psi_{000}\rangle\langle\psi_{000}|), \tag{21}$$

in pure product states, and as usual the optimal EWs can be written as a partial transpose of the following positive operators:

$$^1\mathcal{W}_{\text{opt}}^{(3)^{T_{23}}} = a_1(|100\rangle + |011\rangle)(\langle 100| + \langle 011|),$$

$$^2\mathcal{W}_{\text{opt}}^{(3)^{T_3}} = a_2(|001\rangle + |110\rangle)(\langle 001| + \langle 110|),$$

$$^3\mathcal{W}_{\text{opt}}^{(3)^{T_2}} = a_3(|010\rangle + |101\rangle)(\langle 010| + \langle 101|), \tag{22}$$

respectively. Now the EW $\mathcal{W}_R^{(3)}$ can be written in terms of $^i\mathcal{W}_{\text{opt}}$ as

$$\mathcal{W}_R^{(3)} = a_1{}^1\mathcal{W}_{\text{opt}} + a_2{}^2\mathcal{W}_{\text{opt}} + a_3{}^3\mathcal{W}_{\text{opt}} + 2\omega_2|\psi_{000}\rangle\langle\psi_{000}|$$
$$= a_1{}^{(1)}\mathcal{Q}^{T(23)} + a_2{}^{(2)}\mathcal{Q}^{T(3)} + a_3{}^{(3)}\mathcal{Q}^{T(2)} + 2\omega_2\mathcal{P}. \tag{23}$$

As mentioned in Sec. IV, for $\omega_2 \geq$ the EW $\mathcal{W}_R^{(3)}$ becomes decomposable and cannot detect PPT entangled states.

B. *n* qudit ($d\otimes d\otimes\cdots\otimes d$)

For n particles with the same dimensions the extra terms σ'_S will appear in EWs, provided that the requirement $|S| \leq n-3$ is met. Then the Eq. (9) becomes

$$\mathcal{W}_R^{(n)} = \sum_{S\subsetneq N'} b_S\sigma_S + db_{2,\ldots,n}|\psi_{00\ldots0}\rangle\langle\psi_{00\cdots0}|$$
$$+ \sum_{S\subsetneq N',|S|\leq n-3} a'_S\sigma'_S, \tag{24}$$

where

$$\sigma'_S = \sum_{i_1,\ldots,i_n=0}^{d-1} |i_1\rangle\langle i_1| \otimes O'^{(2)}_{i_2} \otimes \cdots \otimes O'^{(n)}_{i_n},$$

$$O'^{(k)}_{i_k} = \begin{cases} |i_1\rangle\langle i_1|\delta_{i_1 i_k}, & k \in S, \\ |i_k\rangle\langle i_k| \prod_{k\neq m=1}^{n}(1 - \delta_{i_k i_m}), & k \notin S. \end{cases}$$

We discuss below the most simple case: an REW consisting of just two qudits with the same dimension d, that is

$$\mathcal{W}_R^{(2)} = a_1 I_{d^2} + d(a_2 - a_1)|\psi_{00}\rangle\langle\psi_{00}|,$$

where for $a_2=0$ it reduces to the well-known reduction EW (the term "reduction-type EWs" for general $\mathcal{W}_R^{(n)}$ is inspired from this particular case). In this case we have only P_2 which can take values between 0 and 1. So the feasible region is just the line segment $0 \leq P_2 \leq 1$. Setting 0 and 1 in $\mathcal{F}(P_2) = a_1 + (a_2 - a_1)P_2$ yields $a_1 \geq 0$ and $a_2 \geq 0$, respectively. The eigenvalues are a_1 and two $a_2 - a_1$ where the second one must be negative to ensure detecting some entangled states.

C. Three particles ($2\otimes 3\otimes 4$)

As a particular example of REWs with different dimension let us discuss three particles with $2\otimes 3\otimes 4$ dimensions,

$$\mathcal{W}_R^{(3)} = a_1 I_8 + (a_2 - a_1)\sigma_2 + (a_3 - a_1)\sigma_3 + 2(a_{2,3} + a_1 - a_2 - a_3)$$
$$\times|\psi_{000}\rangle\langle\psi_{000}| + a'_1\sigma'_1 + a'_2\sigma'_2 + a'_3\sigma'_3, \tag{25}$$

where we have

$$\sigma'_1 = |012\rangle\langle 012| + |013\rangle\langle 013| + |021\rangle\langle 021| + |022\rangle\langle 022|$$
$$+ |023\rangle\langle 023| + |102\rangle\langle 102| + |103\rangle\langle 103| + |120\rangle\langle 120|$$
$$+ |122\rangle\langle 122| + |123\rangle\langle 123|,$$

$$\sigma'_2 = |002\rangle\langle 002| + |003\rangle\langle 003| + |112\rangle\langle 112| + |113\rangle\langle 113|,$$

$$\sigma'_3 = |020\rangle\langle 020| + |121\rangle\langle 121|.$$

Here all possible $\sigma'_S(S=\{2\},\{3\}$ and $\varnothing)$ can appear. In this case the feasible region lies in a space of dimension six. The coordinates of apexes and relevant pure product states $|\gamma\rangle$ are $(P_2, P_3, P_{23}, P'_1, P'_2, P'_3)$,

$$|\gamma\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \to (0,0,0,0,0,0),$$

$$|\gamma\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \to (1,0,0,0,0,0),$$

$$|\gamma\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \to (0,1,0,0,0,0),$$

$$|\gamma\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \to (1,1,1,0,0,0),$$

$$|\gamma\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \rightarrow (0,0,0,1,0,0),$$

$$|\gamma\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \rightarrow (1,0,0,0,1,0),$$

$$|\gamma\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \rightarrow (0,1,0,0,0,1).$$

Again choosing all combinations of six apexes out of seven, one can find the boundary of the feasible region as

$$P'_1, P'_2, P'_3, P_{2,3} \geqslant 0,$$

$$P_2 - P_{2,3} - P'_2 \geqslant 0,$$

$$P_3 - P_{2,3} - P'_3 \geqslant 0,$$

$$1 + P_{2,3} - P_2 - P_3 - P'_1 \geqslant 0,$$

where the EWs corresponding to these hyperplanes are

$${}^2\mathcal{W}_{\rm opt} = a_2(\sigma_2 - 2|\psi_{000}\rangle\langle\psi_{000}| - \sigma'_2),$$

$${}^3\mathcal{W}_{\rm opt} = a_3(\sigma_3 - 2|\psi_{000}\rangle\langle\psi_{000}| - \sigma'_3),$$

$${}^1\mathcal{W}_{\rm opt} = a_1(I_{24} - \sigma_2 - \sigma_3 + 2|\psi_{000}\rangle\langle\psi_{000}| - \sigma'_1). \quad (26)$$

Taking the partial transposition of ${}^i\mathcal{W}_{\rm opt}$, $i=1,2,3$ with respect to $\{2,3\}\backslash\{i\}$ yields

$${}^1\mathcal{W}_{\rm opt}^{T_{23}} = a_1(|100\rangle + |011\rangle)(\langle100| + \langle011|),$$

$${}^2\mathcal{W}_{\rm opt}^{T_3} = a_2(|001\rangle + |110\rangle)(\langle001| + \langle110|),$$

$${}^3\mathcal{W}_{\rm opt}^{T_2} = a_3(|010\rangle + |101\rangle)(\langle010| + \langle101|), \quad (27)$$

respectively. Evidently these EWs are optimal, since these have been written as the partial transposition of pure maximally entangled states.

VII. BELL-DIAGONAL EWS BY LP METHODS

Recently multiqubit Bell decomposable entangled witnesses (BDEWs) [19] have been introduced as

$$\mathcal{W}_{\rm BD} = \sum_{i_1 i_2 \cdots i_n = 0,1} a_{i_1 i_2 \cdots i_n} |\psi_{i_1 i_2 \cdots i_n}\rangle\langle\psi_{i_1 i_2 \cdots i_n}|, \quad (28)$$

where $|\psi_{i_1 i_2 \cdots i_n}\rangle$ $(d_i=2, i=1,2,\ldots,n)$ are n-qubit maximally entangled orthonormal states, i.e.,

$$|\psi_{i_1 i_2 \cdots i_n}\rangle = (\sigma_z)^{i_1} \otimes (\sigma_x)^{i_2} \otimes \cdots \otimes (\sigma_x)^{i_n} |\psi_{00\cdots0}\rangle, \quad (29)$$

where σ_x and σ_z are usual Pauli matrices.

In general it is hard to find the BDEWs with the feasible region of simplex type or even polygon type, namely those which can manipulated by the LP method. Here we give two examples which are both set in the LP problem, where only one of them (the first example) can be solved exactly by the prescription of this paper.

The first example is EW of the form

$$\mathcal{W}_1 = aI_{2^n} + 2(b-a)|\psi_{00\cdots0}\rangle\langle\psi_{00\cdots0}| + 2(c-a)|\psi_{00\cdots01}\rangle \times \langle\psi_{00\cdots01}| + (d-a)\sigma, \quad (30)$$

where

$$\sigma = I_{2^n} - \{(|\psi_{011\cdots110}\rangle\langle\psi_{011\cdots110}| + |\psi_{11\cdots10}\rangle\langle\psi_{11\cdots10}|) + (|\psi_{011\cdots1}\rangle\langle\psi_{011\cdots1}| + |\psi_{11\cdots11}\rangle\langle\psi_{11\cdots11}|) + (|\psi_{00\cdots0}\rangle \times \langle\psi_{00\cdots0}| + |\psi_{10\cdots0}\rangle\langle\psi_{10\cdots0}|) + (|\psi_{00\cdots01}\rangle\langle\psi_{00\cdots01}| + |\psi_{10\cdots01}\rangle\langle\psi_{10\cdots01}|)\}.$$

The eigenvalues of $\mathcal{W}_1$ are $a, 2b-a, 2c-a, d$. This BDEW is similar to the one introduced in [19], where the extra term σ is added to optimize the EWs corresponding to the boundary plane of the feasible region. Suppose that $P_{00\cdots0} = 2|\langle\psi_{00\cdots0}|\gamma\rangle|^2$, $P_{00\cdots01} = 2|\langle\psi_{00\cdots01}|\gamma\rangle|^2$, and $P = \mathrm{Tr}(\sigma|\gamma\rangle\langle\gamma|)$. Then the pure product states which produce the apexes are

$$|\gamma\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}0\\1\end{pmatrix} \rightarrow (P_{00\cdots0} = 0, P_{00\cdots01} = 0, P = 0),$$

$$|\gamma\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}1\\0\end{pmatrix} \rightarrow (P_{00\cdots0} = 0, P_{00\cdots01} = 0, P = 1),$$

$$|\gamma\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}1\\0\end{pmatrix} \rightarrow (P_{00\cdots0} = 1, P_{00\cdots01} = 0, P = 0),$$

$$|\gamma\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \rightarrow (P_{00\cdots0} = 0, P_{00\cdots01} = 1, P = 0),$$

and consequently these yield the following hyperplanes surrounding the feasible region (see Fig. 2):

$$P_{00\cdots0}, P_{00\cdots01}, \quad P \geqslant 0,$$

$$1 - P_{00\cdots0} - P_{00\cdots01} - P \geqslant 0.$$

The positivity of the last constraint comes from the positivity of the expectation value of the following optimal EW:

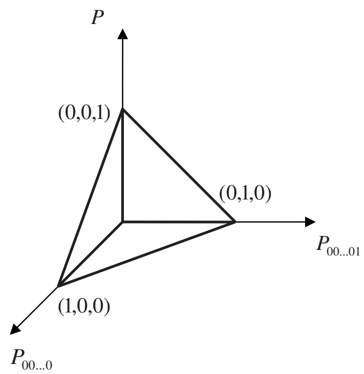

FIG. 2. 3-simplex displaying the feasible region of multiqubit BDEW $\mathcal{W}_1$.

$$\mathcal{W}_{\rm opt} = I - 2|\psi_{00\cdots0}\rangle\langle\psi_{00\cdots0}| - 2|\psi_{00\cdots01}\rangle\langle\psi_{00\cdots01}| - \sigma$$

in pure product states $|\gamma\rangle$, since it can be written as the partial transpose of a positive operator with respect to the first particle as

$$\mathcal{W}_{\rm opt}^{T_1} = 2(|\psi_{11\cdots10}\rangle\langle\psi_{11\cdots10}| + |\psi_{11\cdots11}\rangle\langle\psi_{11\cdots11}|).$$

Now the remaining task is to solve the following LP problem:

$$\text{minimize } a + (b-a)P_{00\cdots0} + (c-a)P_{00\cdots01} + (d-a)P$$

$$\text{subject to } \begin{array}{c} 1 - P_{00\cdots0} - P_{00\cdots01} - P \geqslant 0, \\ P_{00\cdots0}, P_{00\cdots01}, \quad P \geqslant 0. \end{array} \tag{31}$$

Thus, the above problem is reduced to LP and can be solved by the simplex method. Setting the apexes in Eq. (30) we deduce that a,b,c,d should be positive. Now, the operator $\mathcal{W}_1$ fulfills the properties of EWs if at least one of its eigenvalues is negative, namely $2b-a<0$ or $2c-a<0$.

The second example which sets in the LP problem is

$$\mathcal{W}_2 = aI_{2^n} + 2^{n-1}(b-a)|\psi_{00\cdots0}\rangle\langle\psi_{00\cdots0}| + 2^{n-1}(c-a)|\psi_{10\cdots0}\rangle\langle\psi_{10\cdots0}|$$

which cannot be solved by the prescription of this paper. Its feasible region can be determined by Lagrangian multiplier method, as it is discussed in [19]. Let $P_{00\cdots0}=2|\langle\psi_{00\cdots0}|\gamma\rangle|^2$, $P_{10\cdots0}=2|\langle\psi_{10\cdots0}|\gamma\rangle|^2$. There, the problem reduces to

$$\text{minimize } a + 2^{n-2}(b-a)P_{00\cdots0} + 2^{n-2}(c-a)P_{10\cdots0}$$

$$\text{subject to } \begin{cases} \dfrac{1}{2^{n-2}} - P_{00\cdots0} + \left(1 - \dfrac{1}{2^{n-2}}\right)P_{10\cdots0} \geqslant 0, \\ \dfrac{1}{2^{n-2}} - P_{10\cdots0} + \left(1 - \dfrac{1}{2^{n-2}}\right)P_{00\cdots0} \geqslant 0, \\ P_{00\cdots0}, \quad P_{10\cdots0} \geqslant 0. \end{cases} \tag{32}$$

These constraints cannot be reached by partial transposition approach, and the feasible region is estimated by a convex hall of apexes (see Fig. 3). Using LP again, for the positivity of $\mathrm{Tr}(\mathcal{W}_2\rho_s)$ with all mixed separable states a,b,c and $(1-2^n)a+2^{n-1}(b+c)$ must be positive. The eigenvalues of $\mathcal{W}_2$ are

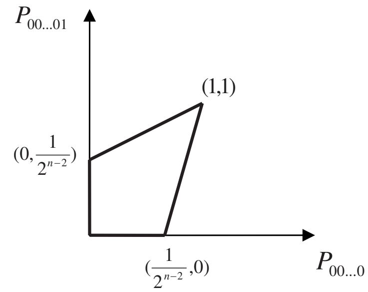

FIG. 3. Convex polygon displaying the boundaries of the feasible region for multiqubit BDEW $\mathcal{W}_2$.

$$\lambda_1 = a,$$

$$\lambda_2 = (2 - 2^{n-1})a + 2^{n-1}b,$$

$$\lambda_3 = 2^{n-1}(c-a). \tag{33}$$

Therefore in order to be $\mathcal{W}_2$ as an EW, at least one of the eigenvalues of λ_2 or λ_3 must be negative.

VIII. DETECTING SOME ENTANGLED STATES BY $\mathcal{W}_R^{(n)}$

This section is devoted to some entangled states which can be detected by general $\mathcal{W}_R^{(n)}$ and three-qubit REW $\mathcal{W}_R^{(3)}$. First we consider some Bell states. All of the Bell states $|\psi_{i\,0\cdots0}\rangle$ can be detected by $\mathcal{W}_R^{(n)}$, since we have

$$\mathrm{Tr}(\mathcal{W}_R^{(n)}|\psi_{00\cdots0}\rangle\langle\psi_{00\cdots0}|) = \omega_2,$$

$$\mathrm{Tr}(\mathcal{W}_R^{(n)}|\psi_{i0\cdots0}\rangle\langle\psi_{i0\cdots0}|) = \omega_1, \quad i \neq 0,$$

therefore for $\omega_2<0$ one can detect $|\psi_{00\cdots0}\rangle$ and for $\omega_1<0$ the other modulated Bell states can be detected by $\mathcal{W}_R^{(n)}$. On the other hand, imposing some constraints on the operator

$$\rho_{i,0,\cdots,0} = \frac{1}{B\,\mathrm{Tr}(\rho_s) + Dd_1}\left\{ B\left(I - \sum_{j=0}^{d_1-1}|\psi_{j0\cdots0}\rangle\langle\psi_{j0\cdots0}|\right) + Dd_1|\psi_{i0\cdots0}\rangle\langle\psi_{i00\cdots0}|\right\}, \tag{34}$$

one can get a density matrix which can be detected by $\mathcal{W}_R^{(n)}$, where ρ_s denotes the separable state inside the parentheses on the right-hand side. The positivity of $\rho_{i,0,\ldots,0}$ constrains B and D to be positive and in order to detect both $\rho_{0,0,\ldots,0}$ and $\rho_{i,0,\ldots,0}, i\neq 0$, we must have

$$B\,\mathrm{Tr}(\mathcal{W}_R^{(n)}\rho_s) + Dd_1\omega_2 < 0, \tag{35}$$

$$B\,\mathrm{Tr}(\mathcal{W}_R^{(n)}\rho_s) + Dd_1\omega_1 < 0, \tag{36}$$

respectively. Because of the positivity of D, Eq. (35) is satisfied if $\omega_2<0$ and Eq. (36) is satisfied if $\omega_1<0$. Now, we can proceed our discussion further to detect PPT entangled

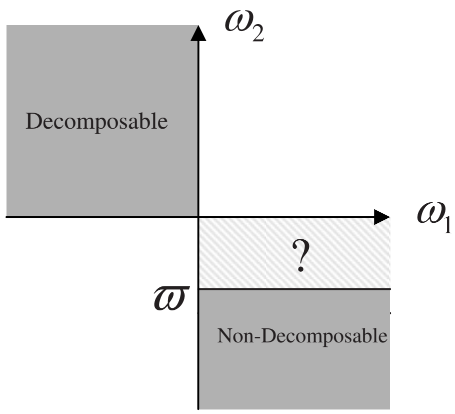

FIG. 4. Decomposable and nondecomposable regions of REWs: for $\omega_2 \geqslant 0$ the REWs are decomposable, for $\omega_2 < \varpi$ the REWs are nondecomposable and in the dashed region, the decomposability or nondecomposability of REWs is still open for debate.

states which is useful for determining the nondecomposable region. Here we do not discuss the decomposability and nondecomposability issues in detail, since it needs the other opportunity and comes from elsewhere. $\rho_{i,0,\ldots,0}$ is PPT states with respect to any subsystem of the particles, if B-D is positive, Eq. (35) yields

$$1 \leqslant \frac{B}{D} < \frac{-d_1\omega_2}{\mathrm{Tr}(\mathcal{W}_R^{(n)}\rho_s)} \Rightarrow \omega_2 < \varpi, \tag{37}$$

where

$$\varpi := -\frac{1}{d_1}\mathrm{Tr}(\mathcal{W}_R^{(n)}\rho_s). \tag{38}$$

The above requirement makes $\rho_{0,0,\ldots,0}$ a class of PPT entangled state which can be detected by $\mathcal{W}_R^{(n)}$. But to detect the other $\rho_{i,0,\ldots,0}$ we must have $\omega_1 < \varpi$ which is impossible. This is in agreement with the discussion made in Sec. V. Now, combining the obtained results with those of Eq. (16) which implies that $\mathcal{W}_R^{(n)}$ is decomposable provided that ω_2 is positive, one can rather determine the decomposability and nondecomposability of the REWs (see Fig. 4).

Furthermore one can construct some entangled states which can be detected by particular REWs. As an example consider entangled density matrices,

$$\rho = \frac{1}{4B+2D}(B\sigma_2 + 2D|\psi_{000}\rangle\langle\psi_{000}|),$$

$$\rho' = \frac{1}{4B+2D}(B\sigma_3 + 2D|\psi_{000}\rangle\langle\psi_{000}|),$$

which can be detected by three-qubit REW (19), with some constraints. The positivity of these states implies that $B, B+2D \geqslant 0$ and the positivity of ρ^{T_3}, ρ'^{T_2} is achieved if $B \pm D \geqslant 0$. In order for $\mathrm{Tr}(\mathcal{W}_R^{(3)}\rho)$ to be negative we should have $B(a_{2,3}+a_2)+D\omega_2<0$, where we have two possibilities: for $D>0$ we have $\omega_2<0$ and for $D<0$ we have $\omega_2>0$.

IX. POSITIVE MAPS

As it is shown in [24], there is a close connection between the positive maps and the entanglement witnesses, i.e., the Jamiołkowski isomorphism

$$d_1 d_2 \cdots d_n (I_{d_1\cdots d_n} \otimes \mathcal{E})|\psi_+\rangle\langle\psi_+| = \mathcal{W}^{(1,2,\ldots,2n)}_{d_1,d_1',\ldots,d_n,d_n'},$$

$$d_i \leqslant d_i', \quad i = 1, \ldots, n, \tag{39}$$

$$\mathcal{E}(\rho) = \mathrm{Tr}_{(1,3,\ldots,2n-1)}[\mathcal{W}^{(1,2,\ldots,2n)}_{d_1,d_1',\ldots,d_n,d_n'}(\rho^T \otimes I_{d_1'd_2'\cdots d_n'})], \tag{40}$$

where

$$|\psi_+\rangle = \frac{1}{\sqrt{d_1 d_2 \cdots d_n}} \sum_{i_1=0}^{d_1-1}\sum_{i_2=0}^{d_2-1}\cdots\sum_{i_n=0}^{d_n-1} |i_1^{(1)} i_1^{(2)}\rangle \times |i_2^{(3)} i_2^{(4)}\rangle \cdots |i_n^{(2n-1)} i_n^{(2n)}\rangle \tag{41}$$

is the maximally entangled state in $\mathcal{H}^{(1)}_{d_1} \otimes \mathcal{H}^{(2)}_{d_1'} \otimes \cdots \otimes \mathcal{H}^{(2n-1)}_{d_n} \otimes \mathcal{H}^{(2n)}_{d_n'}$. Hence the Jamiołkowski isomorphism is a one-to-one mapping between the set of trace preserving quantum operations

$$\mathcal{E}:\mathcal{H}^{(1)}_{d_1} \otimes \mathcal{H}^{(3)}_{d_2} \otimes \cdots \otimes \mathcal{H}^{(2n-1)}_{d_{n_1}} \to \mathcal{H}^{(2)}_{d_1'} \otimes \mathcal{H}^{(4)}_{d_2'} \otimes \cdots \otimes \mathcal{H}^{(2n)}_{d_{n_1}'} \tag{42}$$

and $d_1 \times d_1' \times \cdots \times d_n \times d_n'$ EWs if $d_i \leqslant d_i'$ for $i=1,2,\ldots,n$.

Now, using the Jamiołkowski isomorphism (40), we try to construct the positive maps connected with REWs. Evidently the tensor product of some EWs is also an EW in higher dimension. To be more precise let $\mathcal{W}^{(1,2,\ldots,2n)}_{d_1 d_1' \ldots d_n d_n'}$ be an EW acting on $\mathcal{H}^{(1)}_{d_1} \otimes \mathcal{H}^{(2)}_{d_1'} \otimes \cdots \otimes \mathcal{H}^{(2n-1)}_{d_n} \otimes \mathcal{H}^{(2n)}_{d_n'}$, then depending on possible partition

$$n = n_1 + n_2 + \cdots + n_m, \quad n_i \geqslant 1, \tag{43}$$

one can construct an EW by tensor product of REWs as

$$\mathcal{W}^{(1,2,\ldots,2n)}_{d_1,d_1',\ldots,d_n,d_n'} = \mathcal{W}^{(1,2,\ldots,2n_1)}_{d_1,d_1',\ldots,d_{n_1},d_{n_1}'} \otimes \mathcal{W}^{(2n_1+1,\ldots,2n_1+2n_2)}_{d_{n_1+1},d_{n_1+1}',\ldots,d_{n_1+n_2},d_{n_1+n_2}'} \otimes \cdots \otimes \mathcal{W}^{(2n-2n_m+1,\ldots,2n)}_{d_{n-n_m+1},d_{n-n_m+1}',\ldots,d_n,d_n'}, \tag{44}$$

then using the Jamiołkowski isomorphism (40), one can obtain the corresponding positive map. For instance, considering the tensor product of n REWs (corresponding to the partition $n=1+1+\cdots+1$),

$$\mathcal{W}^{(1,2,\ldots,2n)}_{d_1,d_1',\ldots,d_n,d_n'} = \bigotimes_{k=1}^{n} \mathcal{W}^{(2k-1,2k)}_{d_k,d_k'}$$

with

$$\mathcal{W}^{(2k-1,2k)}_{d_k,d_k'} = a_1^{(2k-1,2k)} I + (a_2^{(2k-1,2k)} - a_1^{(2k-1,2k)})|\psi_{00}^{(2k-1,2k)}\rangle \times \langle\psi_{00}^{(2k-1,2k)}| + a_1'^{(2k-1,2k)}\sigma_1'^{(2k-1,2k)}$$

acting on Hilbert space $\mathcal{H}^{(2k-1)}_{d_k} \otimes \mathcal{H}^{(2k)}_{d_k'}$ with $\sigma_1'^{(2k-1,2k)}$ given in Eq. (11), we get the following positive map:

$$\mathcal{E}^{(n)}(\rho) = \sum_{S\{1,3,\ldots,2n-1\}} \Gamma_S O_S,$$

where

$$\Gamma_S = \prod_{j\in N\setminus S} (a_2^{(2j-1,2j)} - a_1^{(2j-1,2j)}),$$

$$O_S = \bigotimes_{j_i\in S}\left(a_1^{(2j_i-1,2j_i)} I_{d_{j_i}}^{(j_i+1)} + a_1'^{(2j_i-1,2j_i)} \sum_{k=d_{j_i}}^{d'_{j_i}-1} |k\rangle\langle k| \right) \otimes \mathrm{Tr}_{j_1\cdots j_{|S|}} \times(\rho),$$

and $(j_1\cdots j_{|S|})$ is the order of S. Choosing all $a_2=a_1'=0$ and $a_1=1$ yields the generalized reduction map introduced in [23]. As an example for $n=2$ one can easily verify that

$$\mathcal{E}^{(2)}(\rho) = \mathrm{Tr}_{1,3}[\mathcal{W}^{(1,2,3,4)}_{d_1,d_1,d_2,d_2}(\rho^{T_{1,3}} \otimes I^{(2,4)}_{d_1 d_2})] = a_1^{(1,2)} a_1^{(3,4)} \mathrm{Tr}(\rho) I_{d_1}^{(2)} \otimes I_{d_2}^{(4)} + a_1^{(1,2)}(a_2^{(3,4)} - a_1^{(3,4)}) I_{d_1}^{(2)} \otimes \mathrm{Tr}_1(\rho) + a_1^{(3,4)} \times (a_2^{(1,2)} - a_1^{(1,2)}) \mathrm{Tr}_3(\rho) \otimes I_{d_2}^{(4)} + (a_2^{(1,2)} - a_1^{(1,2)})(a_2^{(3,4)} - a_1^{(3,4)})\rho + a_1^{(1,2)} a_1'^{(3,4)} \mathrm{Tr}(\rho) I_{d_1}^{(2)} \otimes \sum_{k=d_2}^{d_2'-1} |k\rangle\langle k| + (a_2^{(1,2)} - a_1^{(1,2)}) a_1'^{(3,4)} \mathrm{Tr}_3(\rho) \otimes \sum_{k=d_2}^{d_2'-1} |k\rangle\langle k| + a_1'^{(1,2)}(a_2^{(3,4)} - a_1^{(3,4)}) \sum_{k=d_1}^{d_1'-1} |k\rangle\langle k| \otimes \mathrm{Tr}_1(\rho) + a_1'^{(1,2)} a_1^{(3,4)} \mathrm{Tr}(\rho) \sum_{k=d_1}^{d_1'-1} |k\rangle \times \langle k| \otimes I_{d_2}^{(4)} + a_1'^{(1,2)} a_1'^{(3,4)} \mathrm{Tr}(\rho) \sum_{k=d_1}^{d_1'-1} |k\rangle\langle k| \otimes \sum_{k=d_2}^{d_2'-1} |k\rangle \times \langle k|.$$

Taking $a_2^{(1,2)}=a_2^{(3,4)}=a_1'^{(1,2)}=a_1'^{(3,4)}=0$ and $a_1^{(1,2)}=a_1^{(3,4)}=1$ yields

$$\mathcal{E}^{(2)}(\rho) = \mathrm{Tr}(\rho) I_{d_1}^{(2)} \otimes I_{d_2}^{(4)} - I_{d_1}^{(2)} \otimes \mathrm{Tr}_1(\rho) - \mathrm{Tr}_3(\rho) \otimes I_{d_2}^{(4)} + \rho \tag{45}$$

and choosing $a_2^{(1,2)}=a_2^{(3,4)}=0$ and $a_1^{(1,2)}=a_1^{(3,4)}=-a_1'^{(1,2)}=-a_1'^{(3,4)}=1$ we reach the new reduction positive map for different dimensions,

$$\mathcal{E}^{(2)}(\rho) = \mathrm{Tr}(\rho) I_{d_1}^{(2)} \otimes I_{d_2}^{(4)} - I_{d_1}^{(2)} \otimes \mathrm{Tr}_1(\rho) - \mathrm{Tr}_3(\rho) \otimes I_{d_2}^{(4)} + \rho - \mathrm{Tr}(\rho) I_{d_1}^{(2)} \otimes \sum_{k=d_2}^{d_2'-1} |k\rangle\langle k| + \mathrm{Tr}_3(\rho) \otimes \sum_{k=d_2}^{d_2'-1} |k\rangle\langle k| + \sum_{k=d_1}^{d_1'-1} |k\rangle\langle k| \otimes \mathrm{Tr}_1(\rho) - \mathrm{Tr}(\rho) \sum_{k=d_1}^{d_1'-1} |k\rangle\langle k| \otimes I_{d_2}^{(4)} + \mathrm{Tr}(\rho) \sum_{k=d_1}^{d_1'-1} |k\rangle\langle k| \otimes \sum_{k=d_2}^{d_2'-1} |k\rangle\langle k|.$$

These examples show that one can construct more positive maps by making a tensor product of EWs $\mathcal{W}_R^{(n)}$ in an arbitrary way provided that the dimensionality condition $d_i \leqslant d_i'$ is satisfied.

X. CONCLUSION

The generalized reduction-type entanglement witnesses with simplex feasible regions are introduced, where the EWs corresponding to hyperplanes surrounding the feasible regions are optimal. These REWs are of types that their manipulation is reduced to the LP problem and can be solved exactly by using the simplex method. As it is shown above, the REWs are decomposable in cases if their second eigenvalue, namely ω_2 becomes positive while for negative values of ω_2, the decomposability or nondecomposability of REWs is still open for debate. Also various other interesting issues remain unsolved, such as keeping the REWs in the realm of the LP problems despite of adding some other operators or entangled states to them.

APPENDIX A

Proof of the inequalities: $0 \leqslant P_S, P_S', P_{2,\ldots,n} \leqslant 1$.

In this appendix we prove that all P_S, P_S', and $P_{2,\ldots,n}$ take the values between 0 and 1. The inequalities $0 \leqslant P_S, P_S' \leqslant 1$ can be easily concluded from the following ones:

$$0 \leqslant \mathrm{Tr}(\sigma_S |\gamma\rangle\langle\gamma|) \leqslant \prod_{k=1}^{n} \sum_{i=0}^{d_k-1} |\alpha_i^{(k)}|^2 = 1,$$

$$0 \leqslant \mathrm{Tr}(\sigma_S' |\gamma\rangle\langle\gamma|) \leqslant \prod_{k=1}^{n} \sum_{i=0}^{d_k-1} |\alpha_i^{(k)}|^2 = 1.$$

For $P_{2,\ldots,n}$, the Cauchy-Schwartz inequality implies that

$$P_{2,\ldots,n} \coloneqq d_1 |\langle \psi_{00\cdots 0}|\gamma\rangle|^2 = \left| \sum_{i=0}^{d_1-1} \alpha_i^{(1)} \alpha_i^{(2)} \cdots \alpha_i^{(n)} \right|^2 = |\langle \alpha^{(1)}|\beta\rangle|^2 \leqslant \||\alpha^{(1)}\rangle\|^2 \||\beta\rangle\|^2,$$

where

$$|\beta\rangle = \begin{pmatrix} \alpha_0^{(2)}\alpha_0^{(3)} \cdots \alpha_0^{(n)} \\ \alpha_1^{(2)}\alpha_1^{(3)} \cdots \alpha_1^{(n)} \\ \vdots \\ \alpha_{d_1-1}^{(2)}\alpha_{d_1-1}^{(3)} \cdots \alpha_{d_1-1}^{(n)} \end{pmatrix},$$

finally using the following inequality:

$$\||\beta\rangle\|^2 = \sum_{i=0}^{d_1-1} |\alpha_i^{(2)}\alpha_i^{(3)} \cdots \alpha_i^{(n)}|^2 \leqslant \prod_{k=2}^{n} \sum_{i=0}^{d_k-1} |\alpha_i^{(k)}|^2 = 1,$$

one can conclude that $0 \leqslant P_{2,\ldots,n} \leqslant 1$.

APPENDIX B

Proof of the feasible region of (15).

In order to obtain the feasible region of (15), we need to evaluate the expectation value of optimal EWs,

$$^{(S)}\mathcal{W}^{(n)}_{\text{opt}} = a_S\Bigg(\sigma_S + \sum_{S\subsetneq S'\neq N'} (-1)^{|S|+|S'|}\sigma_{S'} + d_1(-1)^{|S|+|N'|}|\psi_{00\cdots 0}\rangle\langle\psi_{00\cdots 0}| - \sigma'_S\Bigg), \quad S\subsetneq N' \tag{B1}$$

in pure product states $|\gamma\rangle\langle\gamma|$ where $^{(\emptyset)}\mathcal{W}^{(n)}_{\text{opt}} = {}^{(1)}\mathcal{W}^{(n)}_{\text{opt}}$. Now by taking the partial transpose of $^{(S)}\mathcal{W}^{(n)}_{\text{opt}}$ with respect to (N'/S) we have

$$^{(S)}\mathcal{W}^{(n)^{T_{(N'\setminus S)}}}_{\text{opt}} = a_S \sum_{i\neq j}^{d_1-1} |\Psi^{(S)}_{ij}\rangle\langle\Psi^{(S)}_{ij}|, \tag{B2}$$

where

$$|\Psi^{(S)}_{ij}\rangle \coloneqq |\alpha^{(1)}_{ij}\rangle \otimes |\alpha^{(2)}_{ij}\rangle \otimes \cdots \otimes |\alpha^{(n)}_{ij}\rangle + |\beta^{(1)}_{ij}\rangle \otimes |\beta^{(2)}_{ij}\rangle \otimes \cdots \otimes |\beta^{(n)}_{ij}\rangle, \tag{B3}$$

$$|\alpha^{(k)}_{ij}\rangle = \begin{cases} |i\rangle & \text{if } k\in 1\cup S,\\ |j\rangle & \text{if } k\notin 1\cup S,\end{cases} \qquad |\beta^{(k)}_{ij}\rangle = \begin{cases} |j\rangle & \text{if } k\in 1\cup S,\\ |i\rangle & \text{if } k\notin 1\cup S.\end{cases} \tag{B4}$$

Noting that all of these operators are positive definite and using the relation

$$\text{Tr}(^{(S)}\mathcal{W}^{(n)^{T_{(N'\setminus S)}}}_{\text{opt}}|\gamma\rangle\langle\gamma|) = \text{Tr}(^{(S)}\mathcal{W}^{(n)^{T_{(N'\setminus S)}}}_{\text{opt}}(|\gamma\rangle\langle\gamma|)^{T_{(N'\setminus S)}}) \geq 0$$

yields all feasible regions which are simplexes.

APPENDIX C

Proof of the optimality of $^{(S)}\mathcal{W}^{(n)}_{\text{opt}}$.

Here we try to prove that witness $^{(S)}\mathcal{W}^{(n)}_{\text{opt}}$ is optimal, to this aim we give the proof for the special case $^{(2)}\mathcal{W}^{(n)}_{\text{opt}}$, since the proof of the general case is rather similar to this particular one. According to Ref. [8], the EW $^{(2)}\mathcal{W}^{(n)}_{\text{opt}}$, is optimal if for all positive operator P and $\varepsilon>0$, the following new Hermitian operator

$$\mathcal{W}_{\text{new}} = (1+\varepsilon)^{(2)}\mathcal{W}^{(n)}_{\text{opt}} - \varepsilon P \tag{C1}$$

is no longer an EW. Suppose that there is a positive operator P and $\epsilon\geq 0$ such that $\mathcal{W}_{\text{new}} = {}^{(2)}\mathcal{W}^{(n)}_{\text{opt}} - \epsilon P$ is yet an EW. Let the positive operator P be the pure projection operator $|\psi_i\rangle\langle\psi_i|$, since an arbitrary positive operator can be written as a sum of pure states with positive coefficients as $P = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$.

Now, one should note that the expectation value of the operator $^{(2)}\mathcal{W}^{(n)}_{\text{opt}}$ in pure product states $|\gamma\rangle$ will vanish if they satisfy the following equation:

$$A_iB_j^* + A_jB_i^* = 0 \tag{C2}$$

with

$$A_i = (\alpha_1)_i(\alpha_2)_i, \quad B_j = (\alpha_3)_j(\alpha_4)_j\cdots(\alpha_n)_j.$$

But, it is straightforward to see that, for $A_i, B_j \in \mathbb{R}$, the pure state $|\psi\rangle\langle\psi|$ will be similar to one of the $|\Psi^{(2)}_{ij}\rangle\langle\Psi^{(2)}_{ij}|$ with $i\neq j$ with

$$|\Psi^{(2)}_{ij}\rangle \coloneqq |i\rangle\otimes|i\rangle\otimes|j\rangle\otimes\cdots\otimes|j\rangle + |j\rangle\otimes|j\rangle\otimes|i\rangle\otimes\cdots\otimes|i\rangle, \quad i,j=0,\ldots,d_1-1, \tag{C3}$$

concluding that an arbitrary P has the form $P = \sum_{i\neq j}^{d_1-1} a_{ij}|\Psi^{(2)}_{ij}\rangle\langle\Psi^{(2)}_{ij}|$ with $a_{ij}\geq 0$. Finally, substituting Eq. (C2) in the following expression,

$$\text{Tr}(P|\gamma\rangle\langle\gamma|) = \sum_{ij} a_{ij}|A_iB_j + A_jB_i|^2 = \sum_{ij} a_{ij}\left|\frac{A_j}{B_j}\right|^2 |B_j^*B_i - B_jB_i^*|^2 = 0,$$

and choosing B_i's such that $B_j^*B_i \neq B_jB_i^*$ yields $a_{ij}=0$ and consequently one can conclude that $P=0$.

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